NUMERICAL STUDIES

ON THE ENERGETICS OF SYNOPTIC-SCALE WAVES

BY

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1. Introduction

In the past ten years a large number of numerical experiments has been performed with the aim to simulate the global circulation of the atmosphere. Among other interesting features these calculations show the development of waves in quite a similar way as we observe it in nature. Unfortunately, the dynamics of such phenomena remain for the most concealed in the overall statistics of those models. Therefore it is useful to treat such problems like cyclone development in less sophisticated models.

This we have done with a five layer primitive equation model on the β-plane. We ignored all turbulent transports except that of horizontal diffusion of heat and momentum. Our specific goal was to study the growth of five zonal waves, initially superimposed upon a jet stream-like zonal flow, in relation to parameters defining the lateral profile and the sign of the vertical shear of the zonal flow. Moreover the influence of the static stability is investigated.

2. Energy Budget

The amplification of synoptic-scale disturbances can be caused by two different mechanisms :

a) Kinetic energy of the zonal current (K_Z) is fed into the wavenumber n with kinetic energy K_n. This process is denoted by ($K_Z K_n$). The wavenumber n may also import energy from other waves, this is denoted by ($K_E K_n$). In either case a pure redistribution of kinetic energy within the wave-spectrum takes place. The conversion ($K_Z K_n$) is tightly connected with the barotropic instability. To be effective the zonal current must have a lateral profile such that the gradient of absolute vorticity vanishes somewhere in the region. This kind of instability can also occur in barotropic models.

b) The wavenumber n may grow through a conversion of available potential energy (A_n) into kinetic energy, i.e. through ($A_n K_n$). This is in general the way cyclones in the order of 5000 km are generated in the atmosphere. This thermodynamically induced process (warm rising air - cold sinking air) is associated with baroclinic instability which presumes a non-vanishing vertical wind shear as a necessary condition.

The two statements above are contained in the budget equations for the kinetic and potential energy averaged over the whole volume. They can be written symbolically :

$$\frac{\partial K_n}{\partial t} = (K_E K_n) + (K_Z K_n) + (A_n K_n) + D_n \tag{2.1}$$

$$\frac{\partial A_n}{\partial t} = (A_E A_n) + (A_Z A_n) - (A_n K_n) + G_n \tag{2.2}$$

with D_n as dissipation and G_n as generation. The kinetic energy of the eddies is given by

$K_E = \sum_{n=1}^{\infty} K_n$, that of the mean zonal flow by $K_Z = K_o$.

Applying the rules

$$\sum_1^{\infty} M_n = M_E ; \sum_o^{\infty} M_n = M ; \sum (M_n N_n) = (\sum M_n \sum N_n) \;\; ; \;\; (MM)=0$$

one can derive from (2.1) and (2.2) the familiar expression for the rate of change of K_E, K_Z and A_E, A_Z respectively.

3. Initial Conditions

The initial field is composed of a zonal current $\bar{u}$ (y,p) and a barotropic disturbance characterized by the stream function $\psi'(x,y)$. They assume the form

$$\bar{u}\,(y,p) = 2U_1(1-\frac{p}{p_o}) \left\{(1- \cos \frac{2\pi}{B} y) + q \, (1-\cos \frac{4\pi}{B} y)\right\} \tag{3.1}$$

$$\psi' \,(x,y) = \sum_{n=1}^{5} \frac{Lv_n}{4\pi n} (-1)^n (1- \cos\frac{2\pi}{B} y) \sin\frac{2\pi n}{L} x \tag{3.2}$$

This specification of the initial field allows for a relatively simple determination of $\frac{\partial}{\partial t}(K_Z K_n)_o$ and $\frac{\partial}{\partial t}(A_n K_n)_o$ where the subscript "o" refers to t = 0. These tendencies have been derived analytically on the basis of the quasi-geostrophic equations in order to preassess the development of waves and zonal current (see Fischer et al 1973).

From this method it was possible to gain quite a useful picture about the barotropic and baroclinic conversions that should occur in the numerical model.

The conversion processes are strongly governed by the quantities s^2, β (β: meridional gradient of the coriolisparameter) and q(profile parameter for the horizontal wind). The parameter s^2 is a measure of the static stability and is given by

$$s^2 = -\frac{1}{\rho\Theta} \frac{\partial\Theta}{\partial p} \frac{\pi^2 p_o^2}{f_o^2 B^2} \qquad (3.3)$$

For our experiments the following values have been adopted initially:

s^2 = 0.2, 2.0

q = 0.0, -0.5, -1.0

β = 0.0, 1.5, -1.5 (units $10^{-11} m^{-1} s^{-1}$)

Moreover we have chosen :

L = 12000 km (length of the channel)

B = 6000 km (width of the channel)

fo = 1.22 $10^{-4} s^{-1}$ (coriolisparameter)

v_n = 3 m s^{-1}, K_1 = 20, K_2 = 11, K_3 = 10, K_4=K_5=9 (10^3 Ws/m^2)

The parameter U_1 was prescribed in such a way that for all calculations the initial value of the zonal energy was the same, namely $K_Z=475.10^3$ Ws/m^2 ($K_Z=4/3\ U_1^2\ (3/4\ (1 + q^2)+ q)$).

4. Some Results

As expected, the magnitude of the baroclinic conversions (A_nK_n) was very much dependend upon s^2. Quite a strong reduction in the intensity of (A_nK_n) occurred when s^2 was increased. The second wave (wavelength 6000 km) gained most energy through (A_nK_n), whereas wavenumbers 4 and 5 had no profit at all.

The barotropic conversions (K_ZK_E) were highly affected by the profile parameter q. For 1 = 0 practically no zonal flow energy was transported to the waves, whereas q = -0.5 and even more q = -1.0 yielded a pronounced transfer to the wavenumbers 1 and 2. The remaining waves were damped.

With respect to β, this parameter exerts a strong damping effect on either energy conversions when β was increased from zero to 1.5 10^{-11} $m^{-1}s^{-1}$. There appeared however, an intensification of the eddy generation when β was chosen negative (compared with β = 0).
A summary of the results on the basis of the energy budget (2.1) is displayed in the following table (table 1). There the dissipation D has been evaluated as the residual.

Case 01 $s^2 = 0.5$, $q = -0.5$, $\beta = 0.0$

n	$(K_E K_n)$	$(K_Z K_n)$	$(A_n K_n)$	$\frac{\partial}{\partial t} K_n$	D_n	$\frac{\partial}{\partial t} A_n$
1	- 24	154	240	315	- 55	458
2	- 4	248	340	523	- 61	364
3	- 2	- 66	133	56	- 9	28
4	8	- 18	15	2	- 3	12
5	12	- 8	- 3	- 7	- 8	4
6	10	- 4	5	10	- 1	3
Σ	0	306	730	899	-137	869
0	-306	0	-124	-485	- 55	-1889

Case 03 $s^2 = 0.5$, $q = 0.0$, $\beta = 0.0$

n	$(K_E K_n)$	$(K_Z K_n)$	$(A_n K_n)$	$\frac{\partial}{\partial t} K_n$	D_n	$\frac{\partial}{\partial t} A_n$
1	26	0	260	245	-41	540
2	0	- 3	369	334	-32	273
3	-32	-41	149	68	- 8	35
4	1	-11	3	- 3	4	9
5	1	- 4	- 4	- 9	- 2	3
6	4	- 1	2	5	- 0	2
Σ	0	-61	779	640	79	862
0	61	0	-251	-159	31	-1901

Case 05 $s^2 = 0.5$, $q = -0.5$, $\beta = 1.5 \cdot 10^{-11} m^{-1} sec^{-1}$

n	$(K_E K_n)$	$(K_Z K_n)$	$(A_n K_n)$	$\frac{\partial}{\partial t} K_n$	D_n	$\frac{\partial}{\partial t} A_n$
1	-31	18	64	21	-30	30
2	45	129	50	191	-33	139
3	-16	- 66	160	71	- 7	46
4	-14	- 23	21	- 9	7	3
5	4	- 11	11	- 5	- 9	3
6	12	- 2	0	5	- 5	1
Σ	0	45	306	274	-77	222
0	-45	0	-123	-185	-17	-700

Case 07 $s^2 = 0.5$, $q = 0.0$, $\beta = 1.5 \cdot 10^{-11} m^{-1} sec^{-1}$

n	$(K_E K_n)$	$(K_Z K_n)$	$(A_n K_n)$	$\frac{\partial}{\partial t} K_n$	D_n	$\frac{\partial}{\partial t} A_n$
1	11	- 11	54	14	-40	28
2	16	- 51	119	51	-33	93
3	- 23	- 84	186	69	-10	44
4	- 10	- 20	29	5	6	5
5	2	- 5	- 4	- 14	- 7	1
6	4	- 2	- 2	3	3	1
Σ	0	-173	382	128	-81	172
0	173	0	-175	33	35	-683

Case 11 $s^2 = 0.5$, $q = -0.5$, $\beta = -1.5 \cdot 10^{-4} m^{-1} sec^{-1}$

n	$(K_E K_n)$	$(K_Z K_n)$	$(A_n K_n)$	$\frac{\partial}{\partial t} K_n$	D_n	$\frac{\partial}{\partial t} A_n$
1	30	253	389	520	-152	502
2	-123	142	350	316	- 53	198
3	44	- 46	129	102	- 24	17
4	- 4	- 11	32	2	- 14	24
5	8	- 7	0	- 5	- 7	4
≥6	.45	- 5	- 2	14	- 25	6
Σ	0	326	898	949	-275	751
0	326	0	-107	-527	- 93	-2018

Case 12 $s^2 = 0.5$, $q = 0.0$, $\beta = -1.5 \cdot 10^{-4} m^{-1} sec^{-1}$

n	$(K_E K_n)$	$(K_Z K_n)$	$(A_n K_n)$	$\frac{\partial}{\partial t} K_n$	D_n	$\frac{\partial}{\partial t} A_n$
1	80	72	386	397	-141	624
2	-100	0	389	261	- 27	212
3	- 13	- 6	123	83	- 21	25
4	- 7	- 2	8	- 7	- 6	9
5	- 2	- 3	- 6	- 11	- 1	3
≥6	42	- 3	- 8	8	- 25	4
Σ	0	60	892	731	-221	877
0	60	0	-151	-237	- 26	-2153

Case 02 $s^2 = 2.0$, $q = -0.5$, $\beta = 0.0$

n	$(K_E K_n)$	$(K_Z K_n)$	$(A_n K_n)$	$\frac{\partial}{\partial t} K_n$	D_n	$\frac{\partial}{\partial t} A_n$
1	- 44	265	69	248	-42	94
2	10	122	-14	100	-18	10
3	30	- 8	-14	3	- 6	2
4	2	- 6	- 7	- 14	- 3	1
5	- 1	- 5	- 6	- 14	- 2	0
6	3	- 1	0	1	- 1	0
Σ	0	367	29	324	-72	107
0	-367	0	- 2	-420	-49	-191

Case 04 $s^2 = 0.5$, $q = -1.0$, $\beta = 0.0$

n	$(K_E K_n)$	$(K_Z K_n)$	$(A_n K_n)$	$\frac{\partial}{\partial t} K_n$	D_n	$\frac{\partial}{\partial t} A_n$
1	1	269	61	271	- 60	205
2	-114	494	15	333	- 62	146
3	67	- 44	48	49	- 22	31
4	19	- 32	28	1	- 14	7
5	5	- 9	2	- 10	- 8	1
6	22	- 12	0	15	5	4
Σ	0	666	154	659	-161	394
0	-666	0	14	-779	-127	-736

Table 1 Budgets of kinetic energy averaged from 0 to 150 hours. Units $10^{-3} \frac{N}{m^2}$

The values are averaged over the first 150 hours.

5. The Influence of β^*

It is well known, for example, from the advective model, that increasing the absolute value of β results in a weaker growth rate of unstable waves and stabilizes longer wavelengths. Our result that the sign of β affects the rate of change of eddy kinetic energy has not been stressed in the literature to the author's knowledge. Please remember that there is a difference between the growth rate expressed through the imaginary part of the phase velocity c_i and the rate of change of kinetic energy $\frac{\partial}{\partial t}K_E$. Both quantities are coupled through the relationship:

$$\frac{\partial K_E}{\partial t} = 2kc_i|\hat{\psi}|^2 e^{2kc_i t} = 2kc_i K_E \ ; \ k = \frac{2\pi}{\lambda}, \lambda = \text{wavelength} \qquad (5.1)$$

Although the eigenvalue c_i contains β in quadratic form, which means that the sign of β is irrelevant for this quantity, this might not be true for the eigenfunction $\hat{\psi}$ (complex amplitude of the streamfunction) which also appears on the right-hand side of (5.1). In this case, clearly $\frac{\partial K_E}{\partial t}$ may depend on the sign of β.

Before pursuing this question further it should be emphasized that setting β negative in the basic equations is equivalent to turning the horizontal coordinates by 180°. This means that the sign of $\beta / \frac{\partial \bar{u}}{\partial p}$ is the relevant factor (or more exactly : the gradient of potential vorticity divided by the vertical wind shear).

$$\beta^{*} = \frac{\beta}{k^{2} p_{o} \left(- \frac{\partial \bar{u}}{\partial p}\right)} \tag{5.2}$$

In order to see why $\beta^{*} < 0$ yielded such a different result compared with $\beta^{*} \geq 0$ we made a theoretical study by perturbing the linearized quasi-geostrophic equations around the advective model. Or, in other words, the phase velocity, the stream function and vertical velocity amplitudes were expanded in powers of α^{2}. The latter is defined through

$$\alpha^{2} = - \frac{1}{\rho \Theta} \frac{\partial \Theta}{\partial p} \frac{p_{o}^{2} k^{2}}{f_{o}^{2}} \quad \left(= 4 s^{2} \frac{B^{2} n^{2}}{L^{2}}\right) \tag{5.3}$$

In doing this one obtains a set of inhomogeneous equations, the zero order approximation representing the advective model with its known solution. (The principles of the method are described in McIntyre (1970) on the basis of the Eady model).

Though the y-dependence was ignored in this theoretical study the essential feature of our numerical results could be verified. From zero and first order approximations follows , for example, in non-dimensional form (denoted by an asterisk)

$$\left(\frac{\partial K_E}{\partial t}\right)^* = \frac{C^*_{oi}}{6}\left\{1+\alpha^2\left(\frac{C_{1i}}{C_{oi}} - \frac{1}{15} - \frac{1}{2}\beta^*\right)\right\}$$

$$C_{oi} = \frac{1}{2}\sqrt{\frac{1}{3} - \beta^{*2}} \quad C_{1i} = \frac{1}{4C_{oi}}\left(\frac{1}{6}\beta^{*2} - \frac{2}{4S}\right);$$

$$C_i = C_{oi} + \alpha^2 C_{1i} + --- \qquad (5.4)$$

The above expression demonstrates that the sign of β exerts an effect on the rate of change of kinetic energy in the same way as was observed in our numerical experiments.

Another result is that the vertical transport of geopotential ($\omega\phi$) is affected by the sign of β in such a way that there is a contribution to an upward (downward) transport when $\beta^* > 0$ ($\beta^* < 0$). Thus for $\beta^* < 0$ an accumulation of energy should appear in the lower layers; the opposite should happen for $\beta^* > 0$.

For the simplified theoretical model (no y-dependence, quasi-geostrophic system) the energy budget for a certain level is given by:

$$\left(\frac{\partial K_E}{\partial t}\right)^* = \left(V\phi \right)^* + \left(A_E K_E\right)^* \qquad \left(V\phi\right) = -\frac{\partial}{\partial p}\,\overline{\omega'\phi'}^H \tag{5.5}$$

All terms of (5.5) have been evaluated for $\alpha^2 = 2$ and $\beta^* = 0, \pm 0.5$ by taking the zero and first order approximations only. This approach seems acceptable as can be deduced from fig. 1.

On the other hand, we computed the energy budget evolving from the numerical computation of cases 01,05 and 11 ($q = -0.5$, $s^2 = 0.5$) with merely wavenumber $n = 2$ ($\lambda = 6000$ km) present. The budget equation of the complete model has the form

$$\underline{\frac{K_E}{t}} = \underline{V\phi} + VK_E + \underline{(A_E K_E)} + \underline{(K_Z K_E)} + D_E \tag{5.6}$$

with $VK_E = -\frac{\partial}{\partial p}\,\overline{\omega' K_E}^H$

The essential terms of (5.6)(underlined) with respect to the three cases in question are presented in fig. 2 together with the respective analytical solution represented by (5.5).

There is quite a fair agreement supporting the statements given before about the role of the sign of β^*. This matter is demonstrated too in the following table 2, which shows the rate of change of kinetic energy for cases 03,07 and 12 ($q = 0, s^2 = 0.5$, $n = 1,2,3$) where $(K_Z K_E)$ is small.

One should keep in mind, however, that in our numerical model $\beta^*_{eff} = (\beta - \frac{\partial^2 \bar{u}}{\partial y^2}) / \left[k^2 p_o \left(- \frac{\partial \bar{u}}{\partial p} \right)\right]$ is the relevant parameter instead of β^* in (5.2).

From our assumptions in (3.1) about the structure of the zonal flow it can be deduced that $\beta^*_{eff} \geq \beta^*$ if $\beta^* > 0$ and $\beta^*_{eff} < \beta^*$ if $\beta < 0$.

Thus the modified parameter β^*_{eff} will accentuate the different results of our numerical calculations obtained for positive and negative values of β^* even more.

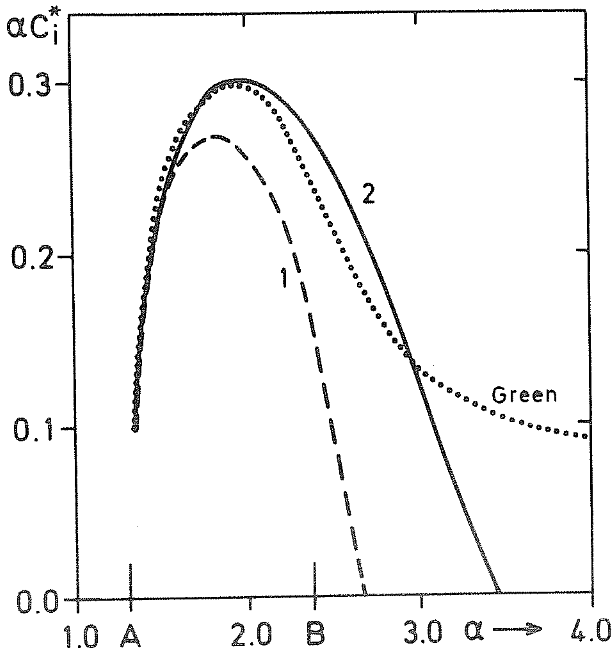
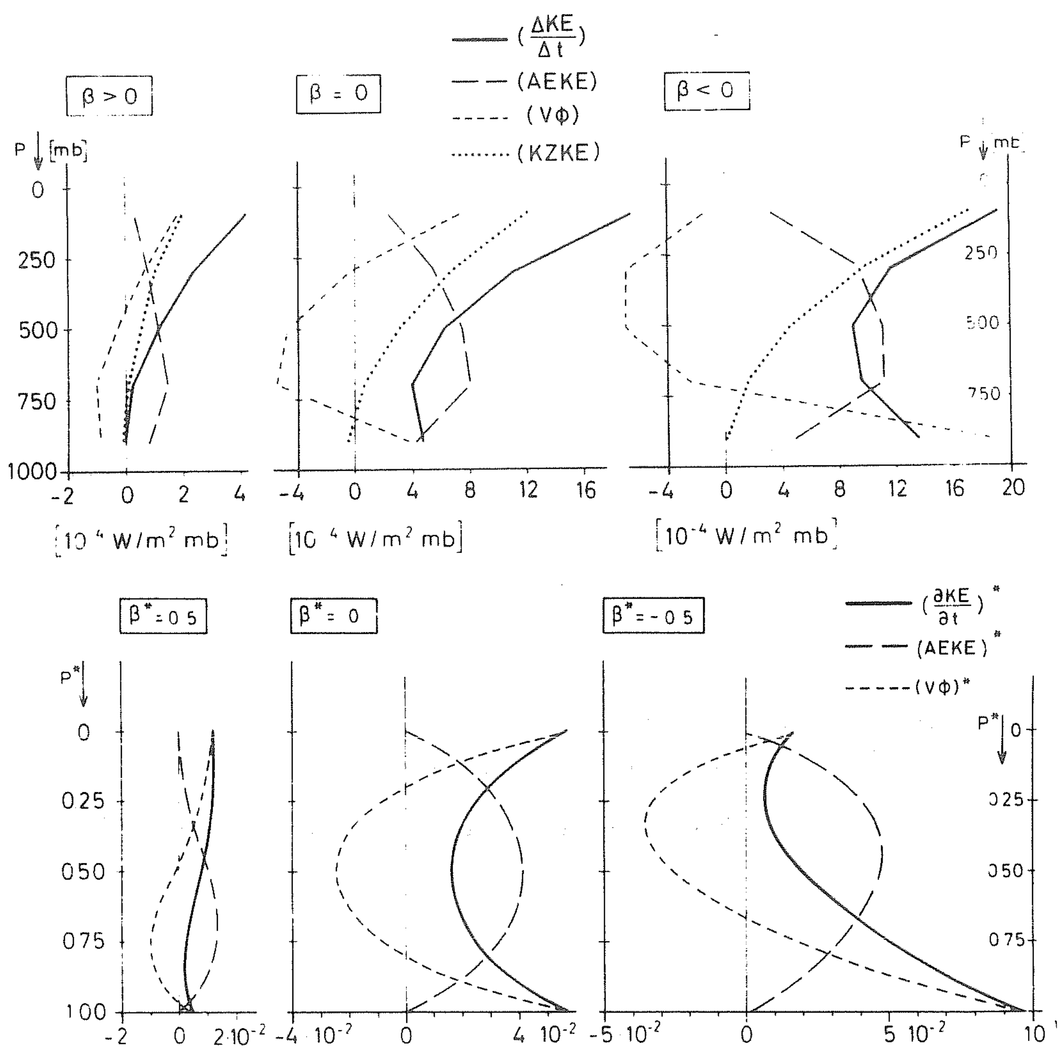


Fig.1
The growth rate αc_i^* as a function of the "wavenumber" α for $\alpha^2 \beta^* = 1$. The numbers 1 and 2 refer to the order of approximation included. The curve derived by Green(1960) has been re-drawn for comparison; A and B denote "neutral wavenumbers" for the advective model and the Eady model respectively (see also McIntyre,1970 page 283).



ig. 2
pper part: Budget terms of kinetic energy for cases 01,05 and 11 (n = 2 λ = 6000 km only) averaged between t = 50 and 100 hours (see 5.6)

ower part: Budget terms for the analytical model with $\alpha^2 = 2$, $\beta^* = 0, \pm 0.5$

Table 2

Case 07	$q = 0$	$\beta = +1.5 \cdot 10^{-11} m^{-1} sec^{-1}$	
n	1	2	3
0-1000mb	15	52	69
900mb	33	0	50
100mb	15	155	139
Case 03	$q = 0$	$\beta = 0$	
n	1	2	3
0-1000mb	245	334	68
900mb	246	293	37
100mb	418	568	162
Case 12	$q = 0$	$\beta = -1.5 \cdot 10^{-11} m^{-1} sec^{-1}$	
n	1	2	3
0-1000mb	398	261	83
900mb	625	467	159
100mb	359	168	39

Rate of change of eddy kinetic energy within 150 hours. Units $10^{-3} W/m^2$ for the values referring to the whole volume and $10^{-6} W/m^2$ mb for those referring to 900mb and 100mb levels.

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