CONSERVATION LAWS, QUASI-TWO-DIMENSIONAL TURBULENCE AND NUMERICAL MODELLING OF LARGE SCALE FLOWS

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1. A SURVEY OF CONSERVATION LAWS

1.1 Conservation of potential vorticity : Ertel's theorem

Let U be absolute velocity, and rot U = grad ξ x grad ζ its curl. The continuity equation in coordinates ξ, ζ, θ reads :

$$\frac{D}{Dt}(\alpha \underline{\mathrm{grad}}\theta . \underline{\mathrm{rot}}U) = \alpha(\underline{\mathrm{grad}}\frac{D\theta}{Dt} . \underline{\mathrm{rot}}U + \underline{\mathrm{grad}}\theta . \underline{\mathrm{rot}}\frac{DU}{Dt}) \qquad (1)$$

Here α is specific volume. If θ is a conservative variable ($D\theta/Dt = 0$) the first term on the right vanishes ; if θ is a thermodynamic variable ($f(\theta, \alpha, P) = 0$, P : pressure), the second term on the right vanishes because, from the equation of motion :

$$\underline{\mathrm{grad}}\theta . \underline{\mathrm{rot}}\frac{DU}{Dt} = \underline{\mathrm{grad}}\theta . (\underline{\mathrm{grad}}P \mathrm{x} \underline{\mathrm{grad}}\alpha) = 0. \qquad (2)$$

If θ is any function of potential temperature, it satisfies both properties and therefore :

$$\frac{D}{Dt}(\alpha \underline{\mathrm{grad}}\theta . \underline{\mathrm{rot}}U) = 0 \qquad (3)$$

The quantity within brackets is known as (Ertel's) potential vorticity. This theorem is central to the dynamics of large scale flows ; it is valid as well for hydrostatic systems, and further, it potentially contains the whole quasi-geostrophic (baroclinic and barotropic) approximation, for the derivation of which it provides the most direct starting point.

1.2 The primitive equations in isentropic coordinates

Isentropic coordinates are seen to arise naturally in the derivation of potential vorticity conservation. A coordinate system (x, y, θ) looks therefore attractive as far as conservation laws are concerned, even though it has scarcely been used up to now because of possible difficulties related to boundary conditions in particular. One advantage of this system is the vanishing of "vertical velocity" $D\theta/Dt$ in the adiabatic case. The hydrostatic (primitive) equations read :

$$\frac{D\underline{V}}{Dt} + f\underline{N} \times \underline{V} + \underline{\text{grad}}\, M = 0 \qquad (4)$$

$$\frac{\partial M}{\partial \theta} - \Pi = 0 \qquad (5)$$

$$\frac{\partial}{\partial t}\frac{\partial P}{\partial \theta} + \text{div}\left(\frac{\partial P}{\partial \theta}\underline{V}\right) = 0 \qquad (6)$$

Here we choose $\theta = c_p T/\Pi$, $\Pi = P^{R/c_P}$; $\underline{N}$ is the vertical unit vector, f Coriolis parameter, M Montgomery's potential or dry static energy ($M = C_p T + gz = \Pi\theta + \phi$, ϕ geopotential); $\underline{V}$ is horizontal velocity and all operators are 2-dimensional. For potential vorticity, we get :

$$D\eta/Dt = 0 \quad \text{with } \eta = (\text{rot } \underline{V} + f)/(\partial P/\partial\theta) \qquad (7)$$

To (4,5,6) we must add boundary conditions. The flow is bounded at top and bottom by surfaces $\theta = \theta_T$ and $\theta_B(x,y,t)$ respectively. We shall use the following symbols : d/dt, D/Dt, $\partial/\partial t$, $\mathscr{d}/\mathscr{d}t$ for derivatives with respect to time, respectively : ordinary derivative, Lagrangian derivative, Eulerian derivative and derivative following a boundary surface. We also denote by $\underline{\mathscr{grad}}$ the gradient following a boundary surface. Boundary conditions then read :

$$(\mathscr{d}/\mathscr{d}t + V.\underline{\mathscr{grad}})\, \theta = 0, \quad P = P_T \quad \text{at } \theta = \theta_T; \qquad (8)$$

$$(\mathscr{d}/\mathscr{d}t + V.\underline{\mathscr{grad}})\, \theta = 0, \quad M = \Pi_B\theta_B + \phi_B \text{ at } \theta = \theta_B. \qquad (9)$$

For deriving the invariants, the following identities will be useful :

$$\mathscr{d}/\mathscr{d}t = \partial/\partial t + \mathscr{d}\theta/\mathscr{d}t \quad \partial/\partial\theta \quad , \tag{10}$$

$$\underline{\mathscr{grad}} = \underline{\mathrm{grad}} + \underline{\mathscr{grad}}\,\theta \quad \partial/\partial\theta \quad , \tag{11}$$

$$\int_x\int_y\int_{\theta_B}^{\theta_T} \begin{pmatrix}\mathrm{div} & \underline{F} \\ \underline{\mathrm{grad}} & F\end{pmatrix} = -\int_x\int_y \begin{bmatrix}\underline{F} \cdot \underline{\mathscr{grad}}\,\theta \\ F \quad \underline{\mathscr{grad}}\,\theta\end{bmatrix}_B^T \quad , \tag{12}$$

$$\frac{d}{dt}\int_{\theta_B}^{\theta_T} f = \int_{\theta_B}^{\theta_T} \frac{\partial f}{\partial t} + \Big[f \; \mathscr{d}\theta/\mathscr{d}t \Big]_B^T \quad , \tag{13}$$

From (6,8,9,12,13) we also get :

$$\frac{d}{dt}\int_X\int_Y\int_{\theta_B}^{\theta_T} - f\frac{\partial P}{\partial\theta} = \int_x\int_y\int_{\theta_B}^{\theta_T} - \frac{\partial P}{\partial\theta}\;\frac{Df}{Dt} \; . \tag{14}$$

1.3 Conservation laws in isentropic coordinates

It is first obvious that the integral with respect to mass of any function of potential enthalpy or potential vorticity are conserved in time. Further, these invariants can be considered independently at each level.

Next, relations (1 to 20) yield successively momentum conservation

$$\frac{d}{dt}\int_x\int_y\int_{\theta_B}^{\theta_T} - \frac{\partial P}{\partial\theta}\;\underline{V} = \int_x\int_y P\;\underline{\mathscr{grad}}\,\phi_B \quad , \tag{15}$$

and energy conservation

$$\frac{d}{dt}\left(\int_x\int_y\int_{\theta_B}^{\theta_T} - \frac{\partial P}{\partial\theta}\left(\frac{V^2}{2} + \frac{M}{1+\kappa}\right) + \int_x\int_y \frac{1}{1+\kappa}\left(\kappa\,\phi_B P_B + \phi_T P_T\right)\right) = 0. \tag{16}$$

1.4 Energy conservation for the linearized system

The linearized form of (4,5,6,8,9) for a flow slightly perturbed around a stratified resting state reads

$$\frac{\partial \underline{V}'}{\partial t} + f\underline{N} \times \underline{V}' + \underline{\text{grad}}\ M' = 0 \tag{17}$$

$$\partial M'/\partial\theta - \Pi' = 0 \tag{18}$$

$$\frac{\partial}{\partial t}\frac{\partial P'}{\partial \theta} + \bar{P}_\theta \ \text{div}\ \underline{V}' = 0 \tag{19}$$

where bars denote mean quantities and primes perturbations. Eliminating everything but M', V' in (17, 18, 19) yields the modified form :

$$\frac{\partial \underline{V}'}{\partial t} + f\underline{N} \times \underline{V}' + \underline{\text{grad}}\ M' = 0\ , \tag{20}$$

$$\left(\bar{P}_\theta^{-1} \frac{\partial}{\partial\theta}\frac{\bar{P}_\theta}{\bar{\Pi}_\theta}\frac{\partial}{\partial\theta}\right)\frac{\partial M'}{\partial t} + \text{div}\ \underline{V}' = 0\ . \tag{21}$$

Imposing the fluctuations in time of boundaries to be small allows a reduction of boundary conditions to fixed boundaries : $\theta = \bar{\theta}_T$ (a constant) and $\theta = \bar{\theta}(Z_B(x,y)) = \bar{\theta}_B(x,y)$ (a function of x and y if we keep finite amplitude mountains). Linearized boundary conditions are :

$$\frac{\partial}{\partial\theta}\frac{\partial M'}{\partial t} = 0 \qquad \text{at } \theta = \bar{\theta}_T \tag{22}$$

$$\left(\theta_B\frac{\partial}{\partial\theta} - 1\right)\frac{\partial M'}{\partial t} + \underline{V}'\cdot\underline{\text{grad}}\ \bar{\phi}_B = 0 \ \text{at}\ \theta = \bar{\theta}_B(x,y) \tag{23}$$

The linear system (20,21,22,23) possesses a quadratic invariant, the perturbation energy

$$E' = \iint_{x\,y}\int_{\theta_B}^{\theta_T} \frac{1}{2}\bar{P}_\theta\left(V'^2 - \Pi_\theta^{-1}\left(\frac{\partial M'}{\partial\theta}\right)^2\right) + \iint_{x\,y}\left(\frac{1}{\bar{\theta}}\frac{\bar{P}_\theta}{\bar{\Pi}_\theta}\frac{M'^2}{2}\right)_B \tag{24}$$

(Notice the boundary term in the available potential energy). One can therefore find a complete system of oscillating eigenfunctions.

The potential vorticity equation survives linearization in the form :

$$\frac{\partial}{\partial t}\left(\text{rot}\ \underline{V}' - \frac{f}{\overline{P}_\theta} \frac{\partial}{\partial \theta} \frac{\overline{P}_\theta}{\overline{\Pi}_\theta} \frac{\partial M'}{\partial \theta} \right) + \underline{V}'.\underline{\text{grad}}\ f = 0 \qquad (25)$$

1.5 The quasi-geostrophic baroclinic approximation

The familiar quasi-geostrophic approximation obtains when considering motion with

a) advective time scale : $U/LT^{-1} \leqslant 1$,

b) small Rossby number : $Ro = U/Lf \ll 1$

which together reduce the equation of motion to geostrophic relation, here $f\underline{N} \times \underline{V} + \underline{\text{grad}}\ M = 0$, to first order in Ro) ;

c) small meridional extent : $L_y/a = O(Ro)$

which yields vanishing divergence to first order in Ro, and therefore a streamfunction $\psi = M/f_0$) ;

d) small density fluctuations : $(\partial P'/\partial\theta)/(\partial\overline{P}/d\theta) = O\ (Ro)$, equivalent to the statement $(RoRi)^{-1} = O\ (Ro)$

(which allows potential vorticity to be linearised). The potential vorticity equation (4) is then rewritten :

$$\left(\frac{\partial}{\partial t} + J(\psi,\ )\right)\left(f+\nabla^2\psi - \frac{f_0^2}{\overline{P}_\theta} \frac{\partial}{\partial \theta} \frac{\overline{P}_\theta}{\overline{\Pi}_\theta} \frac{\partial \psi}{\partial \theta}\right) = 0 \qquad (26)$$

There is no real simplification of boundary conditions, except if we consider small fluctuations only for the boundaries, in which case we get conditions close to (22,23) :

$$\left(\frac{\partial}{\partial t} + J(\psi,\ )\right) \frac{\partial\psi}{\partial\theta} = 0 \qquad \text{at } \theta = \overline{\theta}_T \tag{27}$$

$$\left(\frac{\partial}{\partial t} + J(\psi,\ )\right)\left((\overline{\theta}_B\frac{\partial}{\partial\theta} - 1)\psi + \frac{\overline{\phi}_B}{f_0}\right) = 0 \quad \text{at } \theta = \overline{\theta}_B \tag{28}$$

Here we have used constant $\overline{\theta}_B$ and $\overline{\theta}_T$ as approximations to actual boundaries, which supposes that orography is O (Ro) compared to the scale height. The invariants of the system are the quasi-geostrophic energy

$$\mathfrak{E} = \iint_{x\,y}\int_{\overline{\theta}_B}^{\overline{\theta}_T} - \frac{1}{2}\,\overline{P}_\theta\left((\underline{\text{grad}\psi})^2 - \frac{f_o^2}{\overline{\Pi}}(\partial\psi/\partial\theta)^2\right) + \iint_{x\,y}\left(\frac{f_o^2}{\overline{\theta}}\,\frac{\overline{P}_\theta}{\overline{\Pi}_\theta}\,\frac{\psi^2}{2}\right)_B , \tag{29}$$

the quasi-geostrophic potential enstrophy at any level

$$\mathcal{Z} = \iint_{x\,y} - \frac{1}{2}\,\overline{P}_\theta\left(f + \nabla^2\psi - \frac{f_o^2}{\overline{P}_\theta}\,\frac{\partial}{\partial\theta}\,\frac{\overline{P}_\theta}{\overline{\Pi}_\theta}\,\frac{\partial\psi}{\partial\theta}\right)^2 \tag{30}$$

(or more generally, the integral of any function of potential vorticity) and available potential energy at the top and bottom levels

$$\mathfrak{A}_T = \iint_{x\,y}\left[-\frac{1}{2}\,\overline{P}_\theta\,\frac{\partial\psi}{\partial\theta}^2\right]_T \tag{31}$$

$$\mathfrak{A}_B = \iint_{x\,y}\left[-\frac{1}{2}\,\overline{P}_\theta\left((\frac{\partial}{\partial\theta} - \frac{1}{\overline{\theta}})\,\psi + \frac{1}{\overline{\theta}}\,\frac{\phi_B}{f_0}\right)^2\right]_B \tag{32}$$

1.6 The quasi-geostrophic barotropic approximation

If we consider vertical averages with respect to density, we may get rid of the pure baroclinic dynamics by averaging separately each bracket in (26) and using boundary conditions (27, 28) which yields

$$\left(\frac{\partial}{\partial t} + J(\Psi,\ )\right)\left(f + \nabla^2\Psi + \lambda^2(-\Psi + \frac{\phi_B}{f_o})\right) = 0 \tag{33}$$

where $\boldsymbol{\Psi}$ is the vertical average of Ψ , and

$$\lambda^2 = \frac{f_0^2}{\bar{P}_B - \bar{P}_T} \left(\frac{1}{\bar{\theta}} \frac{\bar{P}_\theta}{\bar{\pi}_\theta} \right)_B \qquad (34)$$

is the inverse squared of the external radius of deformation. The invariants are now reduced to energy

$$\mathfrak{E} = \iint_{x\,y} \frac{1}{2} \left((\underline{\text{grad}}\boldsymbol{\Psi})^2 + \lambda^2 \boldsymbol{\Psi}^2 \right) \qquad (35)$$

and the barotropic form of potential enstrophy

$$\mathfrak{Z} = \iint_{x\,y} \frac{1}{2} \left(f + \nabla^2 \boldsymbol{\Psi} + \lambda^2 \left(-\boldsymbol{\Psi} + \frac{\phi_B}{f_o} \right) \right)^2 \qquad (36)$$

or, more generally, the integral of any function of potential vorticity.

2. NUMERICAL MODELLING OF CONSERVATION LAWS

Almost by definition spectral methods conserve exactly quadratic invariants ; however, more complicated invariants like energy, entropy or potential enstrophy in the general case, are not formally conserved, even though their fluctuations may be small in practice. There is on the contrary in finite difference methods enough degrees of freedom to actually enforce most conservation laws in numerical models.

2.1 The general flux form

Let us consider an advected field q and a density ρ in the general case. The discrete form of the continuity equation is a mass-conserving flux form

$$\frac{\partial \rho_0}{\partial t} + \sum_i F_{0i} = 0 \qquad (37)$$

where index i refers to any neighbour of the gridpoint with index O, and F_{Oi} is the mass flux from O to i ; we require of course that the "neighbour" relation be symmetric, and that $F_{Oi} = -F_{iO}$. The flux form for q which conserves exactly the integral with respect to mass of both q and A(q), has been given by Arakawa

$$\frac{\partial}{\partial t}(\rho q)_0 + \sum_i [q]^A_{0i} F_{0i} = 0 \qquad (38)$$

with

$$[q]^A_{0i} = \delta_{0i}\left(q\frac{dA}{dq} - A\right) / \delta_{0i}\left(\frac{dA}{dq}\right) \qquad (39)$$

where δ_{0i} refers to the finite difference between 0 and i.

2.2 Conservation of energy and potential enstrophy in isentropic coordinates

We shall write the equation of motion in the symbolic form

$$\frac{\partial \underline{V}}{\partial t} + \underline{N}^{X}[\eta \underline{V}] + \underline{\delta}\left(M+\left[\frac{V^2}{2}\right]\right) = 0 \qquad (40)$$

M	u	M	u	M
v	η	v	η	v
M	u	M	u	M

Figure 1.

on Arakawa's C-grid (figure 1) $\underline{\delta} = (\delta_x, \delta_y)$ is the (2-neighbour) finite difference operator, and $\eta = (f+\underline{\delta}x\underline{V})/[\frac{\partial P}{\partial \theta}]$; square brackets denote proper averaging, and $\underline{\mathcal{V}} = (\mathcal{u}, \mathcal{v})$ is the mass flux.

2.2.1 Gradient term

The gradient term vanishes in the vorticity equation ($\underline{\delta}x\underline{\delta} = 0$). As far as this term is concerned, energy conservation holds provided the same linear averaging operator is used for mass fluxes and kinetic energy :

$$\underline{\mathcal{V}} = \left(\left[\frac{\partial P}{\partial \theta}\right]_x u,\ \left[\frac{\partial P}{\partial \theta}\right]_y v \right), \quad \left[\frac{V^2}{2}\right] = \left[\frac{u^2}{2}\right]_x + \left[\frac{v^2}{2}\right]_y \qquad (41)$$

2.2.2 Vorticity term

The simplest averaging schemes we can use for the vorticity term are the following :

$$
\begin{aligned}
R_1 &= \left(-\overline{\overline{\eta}^{y}\,\overline{v}^{y}}^{x} ,\ \overline{\overline{\eta}^{x}\,\overline{u}^{x}}^{y} \right) \\
R_2 &= \left(-\overline{\eta\,\overline{v}^{x}}^{y} ,\ \overline{\eta\,\overline{u}^{y}}^{x} \right) \\
R_3 &= \left(-\overline{\overline{\overline{\eta}^{x}\,v}^{y}}^{x} ,\ \overline{\overline{\overline{\eta}^{y}\,u}^{x}}^{y} \right) \\
R_4 &= \left(-\overline{\overline{\overline{\eta}^{x}}^{y}\,\overline{v}^{y}}^{x} ,\ \overline{\overline{\overline{\eta}^{y}}^{x}\,\overline{u}^{x}}^{y} \right)
\end{aligned}
\tag{42}
$$

and so on : the only rule being that the parity of the number of occurrences of each arithmetic (2-neighbour) average $\overline{}^{x}$ and $\overline{}^{y}$ is kept the same for each quantity η , u, v. It is readily seen from the symmetry properties of the averaging operators that $(R_1+R_3)/2$, R_2, R_4 are energy-conserving. If we consider now potential enstrophy conservation, we get the following relations corresponding to R_1, R_2, R_3, R_4 :

$$\eta \frac{\partial \zeta}{\partial t} + \frac{\eta^2}{2}\ \overline{\overline{\delta_x u + \delta_y v}^{y}}^{x} \sim 0 \tag{43}$$

$$\eta \frac{\partial \zeta}{\partial t} + \frac{\overline{\widetilde{\eta\eta}^{x}}^{y}}{2}\,\delta_x u + \frac{\overline{\widetilde{\eta\eta}^{y}}^{x}}{2}\,\delta_y v \sim 0 \tag{44}$$

$$\eta \frac{\partial \zeta}{\partial t} + \frac{\widetilde{\overline{\eta}^{y}\,\overline{\eta}^{y}}^{x}}{2}\,\delta_x u + \frac{\widetilde{\overline{\eta}^{x}\,\overline{\eta}^{x}}^{y}}{2}\,\delta_y v \sim 0 \tag{45}$$

$$\eta \frac{\partial \zeta}{\partial t} + \frac{\overline{(\overline{\eta}^{y})^2}^{x}}{2}\,\delta_x u + \frac{\overline{(\overline{\eta}^{x})^2}^{y}}{2}\,\delta_y v \sim 0 \tag{46}$$

Here tilda refers to a product between neighbours, and ~ 0 to the vanishing of the sum over gridpoints. Clearly only R_1 is potential enstrophy-conserving, provided

$$\left[\frac{\partial P}{\partial \theta}\right] = \frac{\overline{\overline{\partial P}^{y}}^{x}}{\partial \theta} \tag{47}$$

The best one can do about the three other forms is to combine them in order to get an isotropic average of

$\eta^2/2$, which will therefore factorize out the mass divergence. The combination $R = (R_3 + R_4)/2$ yields

$$\eta \frac{\partial \zeta}{\partial t} + \frac{(\overline{\overline{\eta}}^{y})^{x\,2}}{2} \left(\delta_x u + \delta_y v \right) \sim 0 \tag{48}$$

while $R = 2R_3 - R_2$ leads to

$$\eta \frac{\partial \zeta}{\partial t} + \frac{\widetilde{\widetilde{\eta\eta}}^{\,x}_{\,y}}{2} \left(\delta_x u + \delta_y v \right) \sim 0 \tag{49}$$

the double tilda refering to the arithmetic average of diagonal products. The error on potential enstrophy conservation does not vanish, but gets considerably lower for the two combinations, since usually $\delta_x u + \delta_y v \ll \delta_x u \sim \delta_y v$. Combining energy conservation with such a quasi-conservation of potential enstrophy leads to the following class of vorticity terms :

$$R_\xi = [(1+\xi)(R_1+R_3) - \xi R_2 + (1-\xi) R_4] /3 \tag{50}$$

The absence of an optimum scheme combining both energy- and potential enstrophy exact conservations (at least at this order of averaging in the vorticity term) is rather puzzling. The only such scheme yet found has been worked out for triangular (C-grid) meshes, as reported by Kim. Boundary problems will not be considered in the present context.

2.3 Conservation of energy and moments of potential temperature in sigma coordinates

In the most commonly used sigma system, although simultaneous conservation of energy and moments of potential temperature is no longer straightforward, a formal solution of the problem can nevertheless be worked out. We take finite difference equations in the form

$$\frac{D\underline{V}}{Dt} + f\underline{N}x\underline{V} + \underline{\delta}\phi + [\theta]\,\underline{\delta}\Pi = 0 \qquad (51)$$

$$\frac{\partial}{\partial t}(\theta\delta_\sigma P) + \underline{\delta}\,.([\theta]_A\underline{\mathscr{V}}) + \delta_\sigma([\theta]_A\omega) = 0 \qquad (52)$$

$$\delta_\sigma\phi + [\theta]\delta_\sigma\Pi = 0 \qquad (53)$$

$$\frac{\partial}{\partial t}\delta_\sigma P + \underline{\delta}.\underline{\mathscr{V}} + \delta_\sigma\omega = 0 \qquad (54)$$

where $\sigma = P/P_B$, and $(\underline{\mathscr{V}},\omega)$ is the mass flux $P_B(\underline{V},\dot{\sigma})$; we have supposed $\underline{V},\theta,\Pi,\phi,\underline{\mathscr{V}}$ defined within layers, P and ω at the interfaces. The thermodynamic equation has been used in the general flux from described in section 2.1, corresponding to formal conservation of θ and $A(\theta)$. Successive use of the antisymmetry property of the δ operators leads to guarantee formal energy conservation provided $[\theta] = [\theta]_A$ in the equation of motion and in the hydrostatic equation, and provided

$$\kappa\,\Pi\delta_\sigma P \;\; \theta \; - P\delta_\sigma\Pi \;\; [\theta]_A \;\; \sim \; 0 \qquad (55)$$

which is not true in general. However, if $P_T = 0$ and $A(\theta) = \theta^2$, (55) reduces to

$$\kappa \;\; s \;\; \delta\sigma = \overline{\sigma\delta s}^{\,\sigma} \qquad (56)$$

which defines the optimum location of the levels in terms of $s = \sigma^\kappa$, within layers of given depth $\delta\sigma$.

2.4 Conservation of energy and perturbation energy in sigma coordinates

Again in sigma coordinates, simultaneous conservation of energy and perturbation energy for the linearized equations is no longer straightforward in presence of finite amplitude mountains ; but again a solution can be worked out for this particular problem. This time we shall write the finite difference equations in the form :

$$\frac{D\underline{V}}{Dt} + f\underline{N}\times\underline{V} + \underline{\delta}[\phi'] - [\mathfrak{s}H']\underline{\delta}[\mathfrak{m}] = 0 \qquad (57)$$

$$\delta_\sigma\phi' - \mathfrak{s}H'\delta_\sigma\mathfrak{m} = 0 \qquad (58)$$

$$\frac{\partial}{\partial t}\delta_\sigma P + \underline{\delta}.\underline{\mathscr{V}} + \delta_\sigma\omega = 0 \qquad (59)$$

$$\frac{\partial}{\partial t}(H'\delta_\sigma P) + \underline{\delta}.([H']\underline{\mathscr{V}}) + \delta_\sigma([H']\omega) + (1+\mathfrak{s}H')\left[\frac{D\mathfrak{m}}{Dt}\right]\delta_\sigma P = 0 \qquad (60)$$

with

$$\left[\frac{D\mathfrak{m}}{Dt}\right]\delta_\sigma P = \left[\frac{\partial P}{\partial t} + \omega\right]\delta_\sigma\mathfrak{m} + \left[\underline{\mathscr{V}}.\underline{\delta}\mathfrak{m}\right] \qquad (61)$$

where the basic stratified resting state is specified by $\mathfrak{h}(P)$, $\mathfrak{g}(P)$, $\mathfrak{m}(P)$, respectively its enthalpy, geopotential, Montgomery's potential, and the inverse of static stability

$$\mathfrak{s}(P) = (d\mathfrak{g}/dP)\,(\mathfrak{h}\,d\mathfrak{m}/dP)^{-1} \qquad (62)$$

H', ϕ' refer to the perturbations of enthalpy and geopotential at constant pressure. In (57) to (61), $\underline{V}$, H', $\underline{\mathscr{V}}$, $\mathfrak{s}$, are defined within layers, P, ϕ', ω, $\mathfrak{m}$ at layer interfaces. The linearized equations obtain by dropping the term $\mathfrak{s}H'$ in the thermodynamic equation, and replacing P by basic pressure $\mathfrak{p}$ in the mass fluxes. The energy reads

$$E = \sum\left(\frac{V^2}{2} + H' + [\mathfrak{m}] + \phi'_B\right)\delta_\sigma P, \qquad (63)$$

and the perturbation energy

$$E' = \sum\left[\frac{V'^2}{2} + \mathfrak{s}\frac{H'^2}{2} + \left(-\mathfrak{p}\frac{d\mathfrak{g}}{dP}\frac{P'^2}{2}\right)_B\right]\delta_\sigma\mathfrak{p} \qquad (64)$$

When equations are written in this way,

parallel arguments show that formal conservation holds for both, provided the vertical and horizontal averages used for the horizontal gradient in equation (57) are the adjoint of those used in (61).

3. STATISTICAL MECHANICS OF SIMPLE INVISCID TRUNCATED FLOWS

In this section we consider quasi-geostrophic flows only, with their associated quadratic invariants. The central role played by these invariants is then particularly easy to demonstrate in spectral form, i.e. on the base of the eigenvectors of the Laplacian.

3.1 Statistical equilibrium of barotropic flow without mountains.

This is the simplest case. We consider truncated flow with N real degrees of freedom : the spectral form of energy and enstrophy then read

$$\mathfrak{E} = \sum_{k=1}^{N} K_k \, z_k^2, \quad \mathfrak{Z} = \sum_{k=1}^{N} z_k^2 \tag{65}$$

where K_k is an eigenvalue of $(\lambda^2-\nabla^2)^{-1}$ (finite difference operators are allowed) and z is a real array in one-to-one correspondence with the complex spectral vorticity array .
If N is large enough, the equilibrium state of the system is approximated by Boltzmann's probability law, in the N-dimensional phase space :

$$\mathscr{P}\{z_k\} = \prod_{k=1}^{N} \left(\frac{aK_k+b}{2\pi}\right)^{1/2} \exp -\frac{1}{2}(aK_k+b)\, z_k^2 \tag{66}$$

which is obviously conserved in phase space motion. The corresponding variance reads

$$\langle z_k^2 \rangle = (aK_k+b)^{-1} \tag{67}$$

where parameters a and b are uniquely determined by the given values for $\mathfrak{e}$ and $\mathfrak{z}$. In fact (65) and (67) yield

$$\sum_{k=1}^{N} (N-x(K-K_k))^{-1} = 1 \tag{68}$$

where b has been eliminated, $x = a\mathfrak{z}$ is the unknown and $K = \mathfrak{e}/\mathfrak{z}$. The rational function on the left-hand side has N distinct real poles defining (N-1) intervals, one of which contains zero since $K_{min} \leqslant K \leqslant K_{max}$. It increases within each bounded strictly positive interval and decreases within each bounded strictly negative interval, which yield (M-2) zeros : these have to be discarded since (67) needs to be positive for all k. On the other hand, (68) has N zeros and x = 0 is another solution to be discarded. These exists therefore one and only one solution for x in the interval containing zero, which means one and only one possible equilibrium state. By looking at the derivative of (68) at x = 0, one readily sees that

$$a \leqslant 0 \iff K \geqslant \frac{1}{N} \sum_{k=1}^{N} K_k \tag{69}$$

while a parallel argument exchanging $\mathfrak{e}$ and $\mathfrak{z}$, and K and K^{-1} yields :

$$b \leqslant 0 \iff K^{-1} \geqslant \frac{1}{N} \sum_{k=1}^{N} K_k^{-1} \tag{70}$$

Case (69) (relatively small enstrophy) corresponds to a concentration of energy within larger scales, with a quasi-equipartition of enstrophy at smaller scales ; case (70) (relatively small energy) yields the reverse : concentration of enstrophy at smaller scales, quasi-equipartition of energy at larger scales ; the case in between exhibits quasi-equipartition of energy at smaller scales, quasi-equipartition of enstrophy at larger scales. If one increases resolution for given energy and enstrophy, case (69) eventually holds : in the limit $N \rightarrow \infty$, all the energy gets concentrated in the smallest mode, equipartition of enstrophy holds between all other modes.

Statistical equilibrium spectra are thus seen to be far from realistic because of the inviscid assumption ; however, actual turbulent regimes can be seen as resulting from the frustrated trend of infinite systems towards equilibrium.

3.2 Statistical equilibrium of barotropic flow over mountains

Using the same notations as in 3.1, and denoting by h_k the real array in one-to-one correspondance with topographic vorticity defined in (33), Boltzmann's probability law $\mathscr{P}\{z_k\}$ is proportional to

$$\prod_{k=1}^{N} \exp -\frac{1}{2}\left(aK_k z_k^2 + b(z_k+h_k)^2\right)$$

which after normalization, yields :

$$\mathscr{P}\{z_k\} = \prod_{k=1}^{N}\left(\frac{aK_k+b}{2\pi}\right)^{1/2} \exp -\frac{1}{2}(aK_k+b)\left(z_k+\frac{bh_k}{aK_k+b}\right)^2 \qquad (71)$$

One therefore obtains for mean and variance :

$$\langle z_k\rangle = -\, bh_k(aK_k+b)^{-1} \qquad (72)$$

$$\langle(z_k-\langle z_k\rangle)^2\rangle = (aK_k+b)^{-1} \qquad (73)$$

The statistical mean (72) is also a stationary solution of the deterministic problem as noted by Salmon et al and Bretherton et al, with a streamfunction proportional to potential vorticity $\psi = \frac{b}{a}\eta$. The non-zero mean is the only effect of the presence of mountains : variance itself is not modified. The fact that we obtain a linear form of the general stationary solution $\psi = F(\eta)$ is characteristic of the bias introduced by the truncated spectral approach.

3.3 Statistical equilibrium of baroclinic flows

3.3.1 The simplest two-layer problem

We first consider the simplest case of two equal layers with constant pressure boundary conditions at top and bottom, constant f, and no mountains. Equations (26,27, 28) then yield a system with vertical symmetry. Denoting the internal radius of deformation by λ_I^{-1}, and the average and difference values between the two layers by indices 0 and 1 respectively, we get :

$$\frac{\partial}{\partial t} \Delta_0\psi_0 + J(\psi_0,\Delta_0\psi_0) + J(\psi_1,\Delta_1\psi_1) = 0 \qquad (74)$$

$$\frac{\partial}{\partial t} \Delta_1\psi_1 + J(\psi_0,\Delta_1\psi_1) + J(\psi_1,\Delta_0\psi_0) = 0 \qquad (75)$$

where Δ_0 is the horizontal Laplacian, and $\Delta_1 = \Delta_0 - \lambda_I^2$. Equations (74) and (75) can be understood as the equations for the two first spectral modes in the vertical assuming vertical symmetry, in other words barotropic and first baroclinic modes. The invariants can be restated as :

$$\mathfrak{E} = - < \psi_0\Delta_0\psi_0 + \psi_1\Delta_1\psi_1 > \qquad (76)$$

$$\mathfrak{F} = < (\Delta_0\psi_0)^2 + (\Delta_1\psi_1)^2 > \qquad (77)$$

$$\mathfrak{G} = < \Delta_0\psi_0 \; \Delta_1\psi_1 > \qquad (78)$$

respectively, energy (barotropic plus first baroclinic), enstrophy (again barotropic plus first baroclinic) and correlation between barotropic and baroclinic vorticities.

Boltzmann's law now reads :

$$\mathcal{P}\{z_{0k},z_{1k}\} = \prod_{k=1}^{N} (A_{0k}A_{1k}-c^2)^{1/2} /(2\pi)$$

$$\times \exp-\tfrac{1}{2}(A_{0k}z_{0k}^2+A_{1k}z_{1k}^2+2cz_{0k}z_{1k}) \qquad (79)$$

with

$$A_{0k} = a\ K_{0k} + b,\ A_{1k} = aK_{1k} + b \qquad (80)$$

K_{0k} and K_{1k} referring to the eigenvalues of $-\Delta_0^{-1}$ and $-\Delta_1^{-1}$ respectively (finite difference Laplacians allowed). Straightforward integrations yield :

$$<z_{0k}^2> = A1_k\ (A_{0k}A_{1k} - c^2)^{-1} \qquad (81)$$

$$<z_{1k}^2> = A0_k\ (A_{0k}A_{1k} - c^2)^{-1} \qquad (82)$$

$$<z_{0k}z_{1k}> = 2c\ (A_{0k}A_{1k} - c^2)^{-1} \qquad (83)$$

Like in 3.1, a will be negative if the number of modes is large enough, everything else being equal ; this, and the fact that $K_k^1 \ll K_k^0$ for scales larger than the internal radius of deformation yield a strongly barotropic flow at these scales, in the sense that barotropic energy dominates baroclinic energy, and barotropic enstrophy dominates baroclinic potential enstrophy, while at smaller scales, the flow becomes mixed barotropic - baroclinic.

3.3.2 The multi-layer problem without mountains

Without mountains the vertical direction is separable when one looks for the eigenfunctions of the 3-dimensional Laplacian. We shall then consider L layers and the corresponding vertical eigenvectors Λ_ℓ , $\ell = 1$,L. Like in the 2-layer problem the two boundary invariants disappear because of truncation, and we are left with L + 1 quadratic invariants only : energy and potential enstrophy at each level. Using the same notations as before, we shall consider

$$\mathfrak{E} = K_{\ell k} \; z^2_{\ell k} \tag{84}$$

$$\mathfrak{Z}_\ell = \Lambda_{\ell \ell' \ell''} \; z_{\ell' k} \; z_{\ell'' k} \tag{85}$$

where the summation convention is used ; the invariants $\mathfrak{Z}_\ell$ correspond to the vertical spectrum of potential enstrophy, and $\Lambda_{\ell \ell' \ell''}$ is the discrete integral of the product $\Lambda_\ell \Lambda_{\ell'} \Lambda_{\ell''}$. Boltzmann's law then reads :

$$\mathscr{P}\{z_{\ell k}\} = \alpha \exp -1/2 (a K_{k \ell} \delta_{\ell \ell' \ell''} + b_\ell \Lambda_{\ell \ell' \ell''}) z_{\ell' k} z_{\ell'' k} \tag{86}$$

with summation over all indices on the right hand side ; α is the proper normalizing coefficient. The problem of finding variances for a given k is thus reduced to an eigenvalue problem.

4. CASCADE PROCESSES AND SUBGRIDSCALE MIXING

In pure two-dimensional flows, the tendency for energy to go towards larger scales while enstrophy goes towards smaller scales, can be already foreseen in the trimodal argument of Fjörtöft ; a more general way to look at it is the tendency to absolute equilibria observed within the framework of inviscid truncated dynamics, with the observed equipartitions up and down the spectrum ; in forced real flows where equipartitions are not possible, cascade processes will occur.

4.1 Phenomenology of cascade processes

The characteristic time scale for eddy distortions at scale k^{-1} if we assume homogeneity and isotropy of the statistical ensemble, can be estimated as

$$\tau(k) \sim \left[\int^{k} p^2 E(p)\,dp\right]^{-1/2} \qquad (87)$$

where E(p) is the one-dimensional spectral energy distribution ; here we simply assume that distorsions are essentially produced by larger scales on smaller scales. In turn, the energy transfer rate can be estimated as

$$\varepsilon(k) \sim kE(k) \;/\; \tau(k) \qquad (88)$$

If we look for an inertial range where $\varepsilon(k)$ is constant, i.e. energy is simply transferred across the spectrum by nonlinearity, (87) and (88) yield

$$E(k) \sim \varepsilon^{2/3}\; k^{-5/3} \qquad (89)$$

The transfer in such an energy inertial range will be local : by local we mean that, in the distortion of an eddy of scale k^{-1}, eddies of scale $p^{-1} \lesssim k^{-1}$ dominate. This is readily seen from (87) and (89).

A parallel argument holds for an enstrophy inertial range, where the enstrophy transfer rate

$$\xi(k) = k^3 E(k) \;/\; \tau(k) \tag{90}$$

is supposed constant ; (87) and (90) now yield

$$E(k) \sim \xi^2 \; k^{-3} (\mathrm{Log}\; k/k_o)^{-1/3} \tag{91}$$

and we see that the transfer in such an enstrophy inertial range is nonlocal, i.e. dominated by eddies of scale $p^{-1} \gg k^{-1}$, as seen from (87) and (91).

As stated earlier, we can expect an enstrophy inertial range to develop from the injection scale towards the smaller scales where dissipation occurs, while a backward energy inertial range should develop from the injection scale towards larger scales, if scales large enough are provided for energy to fill in . Note that in the more general case of 2-dimensional turbulence with β-effect and topography, the arguments above are bound to fail in spectral regions where $\tau^{-1}(k)$ is dominated by Rossby wave frequencies or by eigenfrequencies of the topography operator.

4.2 A simple homogeneous turbulence model

The quasi-normal model of homogeneous purely 2-dimensional turbulence reads

$$\left(\frac{d}{dt} + 2\nu k^2\right) Z_{\underline{k}} + 2 \sum_{\underline{p}\underline{q}} (P-Q)\; \Delta_{\underline{kpq}} = 0 \tag{92}$$

$$\left(\frac{d}{dt} + \nu_{kpq}\right) \Delta_{\underline{kpq}} + R_{\underline{kpq}} = 0 \tag{93}$$

with

$$R_{\underline{kpq}} = (\underline{p} \times \underline{q}) \left((P-Q) Z_{\underline{p}} Z_{\underline{q}} + (Q-K) Z_{\underline{q}} Z_{\underline{k}} + (K-P) Z_{\underline{k}} Z_{\underline{p}} \right) \quad (94)$$

$Z_{\underline{k}} = \langle \zeta_{\underline{k}} \ \zeta_{-\underline{k}} \rangle$ are the modal enstrophies, $\Delta_{\underline{kpq}} =$ Re $\langle \zeta_{\underline{k}} \ \zeta_{\underline{p}} \ \zeta_{\underline{q}} \rangle$ are the triple correlations related to triads $\underline{k},\underline{p},\underline{q}$, $\underline{k} + \underline{p} + \underline{q} = 0$, the summation in (92) is over distinct triads, $K = k^{-2}$, and $\nu_{\underline{kpq}} = \nu(k^2+p^2+q^2)$, where ν refers to molecular viscosity. To obtain (92, 93) the hierarchy of moment equations has been truncated at third order by using the Gaussian assumption to get rid of fourth order cumulants. It is well known since Ogura's work that (92, 93) exhibit unrealistic behaviour : in particular, they may yield negative enstrophies. The reason for this behaviour was identified later on : an important effect of the discarded cumulants is to relax triple correlations; thereby producing an irreversibility of the statistical system even in the absence of molecular viscosity ; while (92, 93) is clearly reversible in that case.

This eddy viscosity effect can be recovered using an eddy-damped markovianized form introduced by Orszag. Basically, one has to replace the molecular viscosity term in (93) by an eddy-damping term, with characteristic time scale $\theta_{\underline{kpq}}$ (to be determined) :

$$\left(\frac{d}{dt} + \theta_{\underline{kpq}}^{-1}\right) \Delta_{\underline{kpq}} + R_{\underline{kpq}} = 0 \quad (95)$$

If one further assumes that the evolution time scales for the modal enstrophies are much larger than relaxation time scales, (95) can be replaced by its asymptotic form and combined with (92) to give

$$\left(\frac{d}{dt} + 2\nu k^2\right) Z_{\underline{k}} + 2 \sum \theta_{\underline{kpq}} S_{\underline{kpq}} = 0 \quad (96)$$

$$S_{\underline{kpq}} = (\underline{p}x\underline{q})^2 \ (P-Q) \left((P-Q) Z_{\underline{p}} Z_{\underline{k}} + (Q-K) Z_{\underline{q}} Z_{\underline{p}} + (K-P) Z_{\underline{k}} Z_{\underline{p}} \right) \quad (97)$$

It is readily seen that (96) is energy- and enstrophy-conserving provided $\theta_{\underline{kpq}}$ is symmetric in $\underline{k}$, $\underline{p}$, $\underline{q}$; and that positivity of $Z_{\underline{k}}$ is automatically ensured.

4.3 An estimate for relaxation times

Taking :

$$\theta^{-1}_{\underline{kpq}} = \mu_{\underline{k}} + \mu_{\underline{p}} + \mu_{\underline{q}} , \quad (98)$$

a straightforward estimate for the eddy viscosity $\mu_{\underline{k}}$ is the one extracted directly from (96, 97).

$$\mu_{\underline{k}} = \sum \theta_{\underline{kpq}} \ (\underline{p}x\underline{q})^2 \ (P-Q) \left((Q-K) Z_{\underline{q}} + (K-P) Z_{\underline{p}} \right) \quad (99)$$

However, it has been shown by Kraichnan that (99) overestimates the distortion of small eddies by the larger scales : based on the full equation of motion, it does not separate actual distortion (basically an effect of pressure) from simple sweeping of the eddies by large scale advection. This effect is particularly unwanted in the enstrophy inertial range, where the dynamics is essentially governed by nonlocal effects.

The estimate we give here differs somewhat from Kraichnan's "Test Field" model, although it follows the same line of thought. The pressure effect can be isolated by considering the vorticity and divergence equation in full (here spectral) form. In the absence of pressure, a pure rotational field induces a divergence tendency

$$\frac{d\delta_{\underline{k}}}{dt} = -\sum (\underline{p}x\underline{q})^2 \ PQ \ \zeta_{\underline{p}} \zeta_{\underline{q}} \quad (100)$$

obtained by dropping all terms containing δ in the divergence equation. This germ of divergence in turn reacts on the vorticity field through

$$\frac{d\zeta_{\underline{k}}}{dt} = \sum \underline{k} \cdot \underline{q}\, Q\, \zeta_{\underline{p}} \delta_{\underline{q}} \tag{101}$$

obtained by dropping all terms not containing δ in the vorticity equation. The summations in (100, 101) are not on distinct triads, but on either $\underline{p}$ or $\underline{q}$ (non symmetrized forms). Applying to (100, 101) the same process of markovianized eddy-damping as used in 4.2 yields the modified eddy viscosity form :

$$\mu_{\underline{k}} = \sum \theta_{\underline{kpq}}\, (\underline{p} x \underline{q})^2\, \underline{k} \cdot \underline{q}\, KPQ\, Z_{\underline{p}} \tag{102}$$

again in non symmetrized form, unlike (99). Looking for the influence of very large scales $(P \to \infty)$, we see that the coefficient in (102) remains bounded, while its counterpart in (99) is $0(P)$. More precisely, an estimate of (102) in the isotropic case is

$$\mu_k = \left(\frac{1}{2} \int^k p^2\, E(p)\, dp\right)^{1/2} \tag{103}$$

in the nonlocal limit ; (103) corresponds well to the phenomenological estimate (87).

4.4 A model of the enstrophy inertial range in the isotropic case

In the enstrophy inertial range (96,97,98,102) can be simplified by using the nonlocal assumption $P \gg K \sim Q$: (103) can be used in lieu of (102) and

$$S_{kpq} = \sin^2(\underline{p},\underline{q})\; P/Q\; Z_p (Z_q - Z_k) \tag{104}$$

in lieu of (97) ; all indices now are scalar because of the isotropy assumption and (104) is valid to second order in K/P. Going from discrete summation to continuous integral and using the identity

$$\int_{|\underline{q}-\underline{k}|=p} \sin\hat{K}\, \frac{q^2}{p^3} \Big(F(q)-F(k)\Big)\, dq \equiv \frac{\pi}{2} \frac{\partial}{\partial_{K}{-1}} \frac{\partial F}{\partial K} + O(p) \qquad (105)$$

where $\hat{K}$ is the interior angle opposite to k in the (k,p,q) triangle, yields

$$\left(\frac{d}{dt} + 2\nu k^2\right) Z + \mu \frac{\partial}{\partial_{K}{-1}} \frac{\partial}{\partial K} (Z/2) = 0 \qquad (106)$$

which has to be heuristically modified to

$$\left(\frac{d}{dt} + 2\nu k^2\right) Z + \frac{\partial}{\partial K^{-1}} \frac{\partial}{\partial K} (\mu Z/2) = 0 \qquad (107)$$

in order to recover energy and enstrophy conservation. The enstrophy- and energy fluxes read

$$\Pi_Z = \frac{\partial}{\partial K} (\mu Z/4) \qquad (108)$$

$$\Pi_E = \left(K\frac{\partial}{\partial K} - 1\right) (\mu Z/4) \qquad (109)$$

The energy flux is the sum of two parts : one part directed towards the smaller scales and correlated locally to the enstrophy flux, and another part, directed towards the larger scales, which corresponds to Kraichnan's definition of negative eddy viscosity. Outside the dissipation range, the stationary solution of (106) is $\mu Z = K$, in other words (91) when μ is given by (103) ; in that case the enstrophy flux is constant, and the two parts of the energy flux are exactly opposite to each other. The enstrophy inertial range can therefore be modelled numerically (k bounded by

k_{max}) by using (107, 103) with the boundary condition

$$\Pi_E = 0 \quad \text{at} \quad k = k_{max} \tag{110}$$

4.5 A model for subgridscale mixing in the isotropic case

The model described in 4.4 actually allows numerical modeling in the inertial range. If we want to model subgridscale mixing, we further need a statistical model of the truncated dynamics. Nonlocal expansions again yield

$$\left(\frac{d}{dt}+2\ k^2\right) Z + \frac{\partial}{\partial K^{-1}} \frac{\partial}{\partial K} \left(\tfrac{1}{2}\, \tilde{\mu}\,(k,k_{max})\,(Z(k)-Z(k_{max}))\right) = 0 \tag{111}$$

for dynamics truncated at $k = k_{max}$. The functional form of the "truncated" viscosity $\tilde{\mu}$ is given in Basdevant et al. It departs significantly from μ only in the vicinity of k_{max}, where it goes abruptly to zero. The enstrophy and energy fluxes are given again by (108, 109) with $\tilde{\mu}$ instead of μ. The boundary condition is again (110), but this time it also yields $\Pi_Z(k_{max}) = 0$ because $\tilde{\mu}\ (k_{max}) = 0$: we thus get all the desired conservation properties. Note that an exact solution of the truncated dynamics model (111) is the equipartition of enstrophy between modes. This is a desirable feature, although we have seen that such equipartitions are not in fact governed by nonlocal dynamics : it means that replacing the exact nonlocal dynamics of the truncated model by (111) will not perturb the equilibrium solution.

The model for subgridscale mixing follows by difference, from (107, 111) ; one must however remember that the whole theory suffers from the following severe restrictions:

(i) the assumption of isotropy at all scales has been made; in particular, isotropy of the larger scales is assumed in the expression for eddy viscosity.

(ii) linear terms have been excluded ; in particular, the effect of interactions between Rossby waves and turbulence, and the influence of mountains.

(iii) barotropic flow only has been considered.

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