TECHNICAL MEMORANDUM

19

On the effect of easterly shear on the atmospheric waves in the equatorial zone

X. Y. Du and Z. D. Zhou

Research Department

September 1980

European Centre for Medium-Range Weather Forecasts
Europäisches Zentrum für mittelfristige Wettervorhersage
Centre européen pour les prévisions météorologiques à moyen

In recent years, the normal mode method has been introduced into meteorology. By this method, the objective analysis, initialization and numerical prediction are treated by the same mathematical procedure. By the method of separating variables, the three-dimensional weather forecasting problem may, in general, be treated as a two-dimensional one (in the horizontal) multiplied by a one-dimensional in the vertical. Since the one-dimensional problem in the vertical is more or less easy to manipulate, the investigators have been concentrating their attention on the two-dimensional forecasting problem. In this case, the governing equations are the primitive ones for the barotropic atmosphere. To use the normal mode method, these primitive equations are first linearized with respect to some basic state of atmospheric motion.

When the basic state of atmosphere is at rest, i.e. the mean zonal velocity U=0, the linearized equations may be solved analytically, and Hough functions are their normal modes. These normal modes consist of three kinds of waves: the westward propagating Rossby wave, westward propagating inertia-gravity wave and eastward propagating inertia-gravity wave.

In the basic state of the atmosphere, however, the mean zonal wind component is not zero and may be an arbitrary function of latitude. In this case, the linearized equations have no analytical solutions, and their normal modes can only be obtained in numerical form. The question of how many kinds of waves are contained in normal modes remains open. The numerical results show that in addition to the three abovementioned kinds of waves, there also exists a fourth kind of wave - eastward propagating Rossby waves. This fourth wave sometimes appears more evident when a higher-order difference scheme is used and the truncation error is further reduced. Therefore, it seems to us that there are not sufficient grounds to consider all the eastward propagating Rossby waves simply as computational modes.

To clear up this question, one must solve the linearized equations analytically. Unfortunately, as we mentioned above, the original linearized equations with arbitrary mean zonal wind profile have no analytical solutions. Thus, it is quite desirable to obtain analytical solutions for these equations even in their over-simplified version.

In this paper the zonal wind shear is considered as constant and denoted by s. The equations are written on an equatorial β-plane, and the solutions are applicable only to the equatorial zone, where y is small. We will take the effective height of the atmosphere $H=10^4$m, then the gravity-wave velocity $c=\sqrt{gH}=313\ ms^{-1}$. In the equatorial zone we take $\beta=2.28\cdot 10^{-11} s^{-1} m^{-1}$.

With these assumptions in mind, we may write the linearized equations in their non-dimensional form as follows:

$$\frac{\partial u}{\partial t} + s(y-y_c)\frac{\partial u}{\partial x} + v(s-y) + \frac{\partial \phi}{\partial x} = 0 \tag{1}$$

$$\frac{\partial v}{\partial t} + s(y-y_c)\frac{\partial v}{\partial x} + yu + \frac{\partial \phi}{\partial y} = 0 \tag{2}$$

$$\frac{\partial \phi}{\partial t} + s(y-y_c)\frac{\partial \phi}{\partial x} - s(y-y_c)\,yv + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{3}$$

Here the time unit is taken as $(C\beta)^{-\frac{1}{2}} = 1.18\cdot 10^4$s, and horizontal length unit $C^{\frac{1}{2}}\beta^{-\frac{1}{2}} = 3.7\cdot 10^6$m. Then s = 0.07 corresponds to a mean wind shear of 6ms^{-1} per 10^3km. At the mean position of the subtropical ridge $y = y_c = 0.9$ the mean zonal wind is equal to zero, and on the equator (y = 0) it has a maximal value.

Assume the solutions are of the wave form $\exp(i\omega t+ikx)$, then their amplitudes satisfy the following equations:

$$i\omega u + iks(y-y_c)u + (s-y)v + ik\phi = 0 \tag{4}$$

$$i\omega v + iks(y-y_c)v + yu + \frac{d\phi}{dy} = 0 \tag{5}$$

$$i\omega\phi + iks(y-y_c)\phi - sy(y-y_c)v + iku + \frac{dv}{dy} = 0 \tag{6}$$

Eliminating u and ϕ we get the equation for v as follows:

$$\frac{d^2v}{dy^2} + (a_o + a_1 y)\frac{dv}{dy} + (b_o + b_1\, y - y^2)v = 0 \tag{7}$$

Where

$$a_o = -\frac{2k\omega s}{\omega^2-k^2}$$

$$a_1 = sy_c$$

$$b_o = \omega^2 - k^2 + \frac{k}{\omega} + (1 - 2k\omega + \frac{k^2}{\omega^2})sy_c$$

$$b_1 = (1 + 2k\omega - \frac{2\omega^2}{\omega^2-k^2} - \frac{k^2}{\omega^2})s$$

If we set s=0, our equation (7) will degenerate exactly into Matsuno's equation (Matsuno, 1966). For a derivation of (7) including the effect of orography, see appendix I.

In (7) the parameter s is considered as a small quantity, and only the first order of magnitude is retained. The small terms of higher orders including sy^2 are omitted, since we are interested only in the equatorial zone.

By the following transformations

$$w = v \exp\left(\frac{2+a_1}{4} y^2 + \frac{a_o - b_1}{2} y + \nu\right)$$

$$z = \sqrt{2}y - \frac{\sqrt{2}}{2} b_1$$

equation (7) can be written in the normal form:

$$\frac{d^2w}{dz^2} - z\frac{dw}{dz} + \frac{2b_o - 2 - a_1}{4} w = 0 \qquad (8)$$

In the transformation of v into w, there appears an arbitrary constant ν.

Since we are considering wave motions near the equator, i.e. $y \simeq 0$ then the boundary conditions

$$w \to 0, \quad \text{when } z \to \pm\infty \qquad (9)$$

may be adequate, in the real atmosphere there is an upper limit to $|z|$, and the positions of the poles and the boundary conditions should be quite different ones. However, we take it for granted that these approximations have little effect on the solutions of the lower modes, so also did other investigators.

The equation (8) with boundary conditions (9) poses an eigenvalue problem. The conditions (9) are satisfied only when the constant $\frac{b_o - 1}{2} - \frac{a_1}{4}$ is equal to a positive integer or zero:

$$2b_o - 2 - a_1 = 4n$$

$$(n = 0, 1, 2 \ldots\ldots)$$

Thus, we have the following eigenvalue equation

$$\omega^4 - 2ksy_c\omega^3 - (k^2 + 2n + 1 - \frac{sy_c}{2})\omega^2 + k\omega + k^2 sy_c = 0 \qquad (10)$$

This equation gives a relation between the frequency and the longitudinal wave number for some definite meridional mode. It expresses the dispersion relation in our discussion.

Since (10) is an algebraic equation of the fourth order in ω, we have four roots when n and k are specified. These four roots are denoted by ω_i, i=1, 2,3,4, and the corresponding phase velocity by $-\frac{\omega_i}{k}$.

Equation (10) is solved numerically. It is expected that two of the four roots ω_1 and ω_2 correspond to the two inertia-gravity waves, one of which is propagating westward and the other eastward. The other two roots ω_3 and ω_4 correspond to westward and eastward propagating Rossby waves.

Numerical results obtained under various values of n and k are given in Table 1.

Table 1

ω	k \ n	0	1	2	3	4
ω_1	1	1.04	1.92	2.42	2.82	3.17
	2	2.12	2.61	3.01	3.35	3.65
	3	3.19	3.52	3.82	4.09	4.35
	4	4.26	4.51	4.74	4.96	5.17
ω_2	1	-1.55	-2.05	-2.46	-2.82	-3.14
	2	-2.28	-2.65	-2.98	-3.28	-3.55
	3	-3.11	-3.39	-3.66	-3.89	-4.14
	4	-3.99	-4.21	-4.43	-4.63	-4.83
ω_3	1	0.692	0.308	0.217	0.172	0.144
	2	0.515	0.384	0.313	0.268	0.236
	3	0.433	0.376	0.335	0.304	0.280
	4	0.388	0.358	0.334	0.313	0.296
ω_4	1	-0.0567	-0.0522	-0.0488	-0.0461	-0.0439
	2	-0.101	-0.0948	-0.0899	-0.0858	-0.0823
	3	-0.132	-0.126	-0.121	-0.116	-0.113
	4	-0.153	-0.148	-0.144	-0.140	-0.136

Having obtained the eigenvalues of equation (10), we may now investigate the solutions of (8). The eigenfunctions of (8) are parabolic cylinder functions $D_n(z)$:

$$D_n(z) = (-1)^n \exp\left(\frac{z^2}{4}\right) \frac{d^n}{dz^n} \exp\left(-\frac{z^2}{2}\right)$$

$$D_o(z) = \exp\left(-\frac{z^2}{4}\right)$$

$$D_1(z) = z \exp\left(-\frac{z^2}{4}\right)$$

$$zD_n(z) = D_{n+1}(z) + n\, D_{n-1}(z)$$

and so on.

The solution of (4) – (6) for the amplitude may be written in the forms:

$$v = \exp\left(-\nu - \frac{a_o}{2}y - \frac{a_1}{4}y^2\right) D_n\left(\sqrt{2}y - \frac{\sqrt{2}}{2}b_1\right),$$

$$u = \frac{i}{k(1-\alpha^2)} \exp\left(-\nu - \frac{a_o}{2}y - \frac{a_1}{4}y^2\right) \left\{\left[\left(\frac{\omega}{k} - \frac{a_1}{2} - 1\right) y - \alpha s + \frac{b_1-a_o}{2}\right] D_n\left(\sqrt{2}y - \frac{\sqrt{2}}{2}b_1\right) + \sqrt{2}\, n\, D_{n-1}\left(\sqrt{2}y - \frac{\sqrt{2}}{2}b_1\right)\right\}$$

$$\phi = \frac{i}{k(1-\alpha^2)} \exp\left(-\nu - \frac{a_o}{2}y - \frac{a_1}{4}y^2\right) \left\{\left[\frac{\omega}{k}s(y-y_c)y + \frac{\alpha a_1}{2}y + s + (\alpha-1)y - \alpha\frac{b_1-a_o}{2}\right] D_n\left(\sqrt{2}y - \frac{\sqrt{2}}{2}b_1\right) - \sqrt{2}n\, \alpha D_{n-1}\left(\sqrt{2}y - \frac{\sqrt{2}}{2}b_1\right)\right\}$$

Where $\alpha = \frac{\omega}{k} + s(y-y_c)$

Notice that, in the adjacent ridge and trough pressure systems, u and ϕ have the same phase, but v has a phase lag $\frac{\pi}{2}$. These synoptical facts are reflected by the introduction of imaginary unit i in the above expressions.

In the following calculation we will take $i(\omega^2-k^2) = \exp(-\nu)$.

Multiplying amplitude v by sinkx, and u and ϕ by coskx, we may obtain the wind and geopotential fields for various waves. The pattern of eastward propagating Rossby wave in case of n=k=1 are shown in Fig. 1, wind vectors are represented by arrows. Lengths of the arrows are proportional to the wind speed. Iso-geopotential lines are denoted by solid lines. The south-north extension in Fig. 1 ranges from y = -0.36 to 0.36, and the east-west extension represents one wave length. The eastward propagating Rossby wave is peculiar to equatorial zones, and quite different from that of high and middle-latitudes. Fig. 1 shows that the wave amplitudes tend rapidly to zero from the equator in both directions. Fairly large geostrophic deviations appear in this figure. It is interesting to note that in the northern half of the equatorial low (high) the circulation is clockwise (anti-clockwise). The reverse situation is observed in their southern halves.

Attention must be paid to the situation when s approaches zero. In this case, one of the four roots of equation (10) is zero: $\omega_4 = 0$. Thus, in addition to the three solutions obtained by Matsuno, we also have here an excessive solution $\omega_4 = 0$. At first sight this is a paradox. But in fact, it is quite reasonable. For if $\omega = 0$ is put into the amplitude equations (4) ~ (6) with s = 0, it is found that all amplitudes equal zero and the equations are also satisfied. Therefore $\omega_4 = 0$ is a trivial solution to equations (4) ~ (6) in case of s = 0. In general, $s \neq 0$ we have $\omega_4 \neq 0$ and an eastward propagating Rossby wave is obtained and all its amplitudes are finite. The conclusion follows that eastward propagating Rossby waves are caused by the easterly wind shear. Its amplitudes and frequency approach zero as the shear approaches zero.

The easterly wind shear also modified other kinds of existing waves, both frequencies and amplitudes. But these modifications are only of quantitative character, and qualitatively these waves remain the same as in the case s = 0 obtained by Matsuno. So the results are not given here.

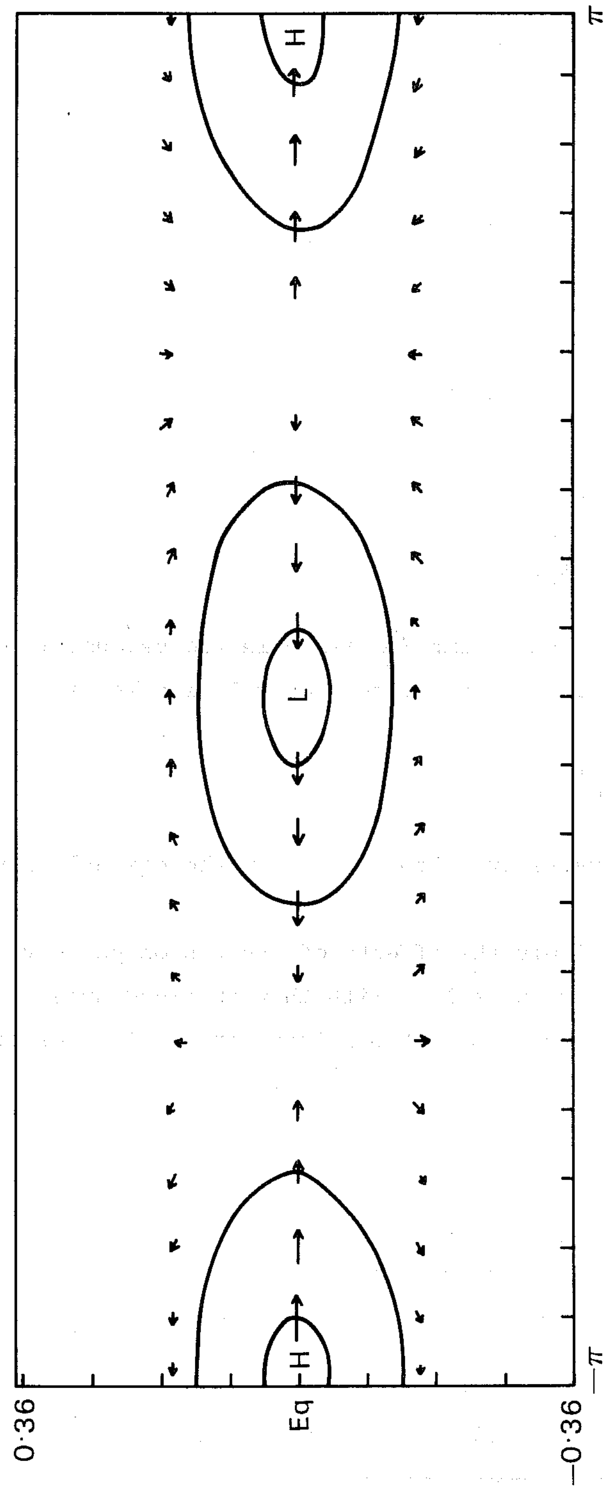

Fig. 1 Easterly propagating Rossby wave.

In addition to the above-mentioned, there exists another kind of wave, the Kelvin waves. The meridional wind component in a Kelvin wave equals zero $v=0$. To obtain the Kelvin wave, we must first put $v=0$ in equations (4) ~ (6), and obtain

$$\left[\omega + ks(y-y_c)\right] u+k\phi = 0 \tag{4a}$$

$$yu + \frac{d\phi}{dy} = 0 \tag{5a}$$

$$ku + \left[\omega + ks(y-y_c)\right] \phi = 0 \tag{6a}$$

From (4a) and (6a) we get

$$-\frac{k\phi}{u} = -\frac{ku}{\phi} = \omega + ks(y-y_c)$$

Substitution into (5a) gives

$$\phi = u=\text{const} \exp\left(-\frac{y^2}{2}\right)$$

The other alternative solution $\exp\left(\frac{y^2}{2}\right)$ is neglected because it does not satisfy boundary conditions. The dispersion relation is as follows

$$\omega = -k - ks\ (y-y_c)$$

Evidently, the Kelvin waves are also altered by the easterly wind shear.

To illustrate quantitatively the effects of shear s on phase velocity, we have compared the figures in Table 1 with that obtained under s = 0, and also calculated the phase velocities of the Kelvin waves. The results are presented in Table 2 and Table 3 for the case n = k = 1

Table 2 for case n = k = 1

Kinds of waves	phase velocity		
	s = 0	s = 0.07	change
W.G.	-1.86	-1.92	-0.06
E.G.	2.11	2.05	-0.06
W.R.	-0.25	-0.31	-0.06
E.R.	0	0.06	0.06
Kelvin	1	0.937	-0.063

Table 3 for case n = 0, k = 1

Kinds of waves	phase velocity		
	s = 0	s = 0.07	change
W.G.	-1.00	-1.04	-0.04
E.G.	1.62	1.55	-0.07
W.R.	-0.62	-0.69	-0.07
E.R.	0	0.07	0.07
Kelvin	1	0.937	-0.063

and for the case n = 0, k = 1 respectively. From these Tables we can see that wind shear s decreases the phase velocity with the exception of E.R. In spite of the large difference in phase speed for various waves, the phase speed changes caused by shear s are nearly of the same magnitude for various waves.

Acknowledgements

Useful comments on this paper have been given by Dr. L. Bengtsson, Dr. K.R. Johannessen, Dr. S. Tibaldi, Prof. T.C. Yeh and Prof. F.K. Chuang, to whom the authors are much obliged.

References

Matsuno, T., 1966 Quasi-geostrophic motions in the equatorial area. J. Met. Soc. Japan, 44, 25-43.

DERIVATION OF EQN. (7) INCLUDING OROGRAPHIC EFFECTS

Let h be the mountain height, then the equation of continuity in its linearized dimensionless version will take the form:

$$\frac{\partial\phi}{\partial t} + s(y-y_c)\frac{\partial\phi}{\partial x} - s(y-y_c)yv + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} - s(y-y_c)\frac{\partial h}{\partial x} = 0$$

Assume the solutions are of the wave form exp($i\omega t+ikx$), and the mountain height of the form exp(ikx). Then at some initial moment, i.e., t=0, their amplitudes satisfy the following equations:

$$i\omega u + iks(y-y_c)u + (s-y)v + ik\phi = 0 \qquad (A\ 1)$$

$$i\omega v + iks(y-y_c)v + yu + \frac{d\phi}{dy} = 0 \qquad (A\ 2)$$

$$i\omega\phi + iks(y-y_c)\phi + iku - s(y-y_c)yv + \frac{dv}{dy} - iks(y-y_c)h = 0 \qquad (A\ 3)$$

Eliminating u from (A 1) and (A 2), we obtain:

$$\left[\omega + ks(y-y_c)\right]\frac{d\phi}{dy} - ky\phi + i\left[\omega^2+2\omega ks(y-y_c) + y(s-y)\right] v = 0 \qquad (A\ 4)$$

Eliminating u from (A 1) and (A 3), we obtain:

$$i\left[\omega^2-k^2+2\omega ks(y-y_c)\right]\phi - \left[(s-y)k + \omega sy(y-y_c)\right] v +$$

$$+ \left[\omega+ks(y-y_c)\right]\frac{dv}{dy} - i\omega ks(y-y_c)h = 0 \qquad (A\ 5)$$

Differentiating (A 5) with respect to y, we have:

$$\left[\omega+ks(y-y_c)\right]\frac{d^2v}{dy^2} + \left[ky-\omega sy(y-y_c)\right]\frac{dv}{dy} + \left[k-\omega s(2y-y_c)\right]v -$$

$$- i\omega ks(y-y_c)\frac{dh}{dy} - i\omega ksh + i\left[\omega^2+2\omega ks(y-y_c)-k^2\right]\frac{d\phi}{dy} +$$

$$+ 2i\omega ks\phi = 0 \qquad (A\ 6)$$

Now, from (A 4) and (A 6) we may eliminate $\frac{d\phi}{dy}$ and obtain:

$$\left[\omega^2+2\omega ks(y-y_c)\right]\frac{d^2v}{dy^2} + \left[\omega ky+(k^2-\omega^2)sy(y-y_c)\right]\frac{dv}{dy}$$

$$- i\omega^2 ks(y-y_c)\frac{dh}{dy} - i\omega^2 ksh +$$

$$+ \left[\omega k-\omega^2 sy + (\omega^2-k^2)(\omega^2+sy_c-y^2) + 2\omega ks(y-y_c)\ (2\omega^2-k^2-y^2)\right] v +$$

$$+ i\left[2\omega^2 ks + (\omega^2-k^2)ky + 2\omega k^2 sy(y-y_c)\right]\phi = 0 \qquad \text{(A 7)}$$

And, at last, eliminating ϕ from (A 5) and (A 7), we have:

$$\frac{d^2v}{dy^2} + (a_o+a_1y)\frac{dv}{dy} + (b_o+b_1y-y^2)v +$$

$$+ iks(y-y_c)\frac{dh}{dy} - i\,\frac{k^2y_c sy}{\omega}\,h = 0.$$

The last two terms on the left hand side reflect the mountain effect.