RESEARCH CONCERNING THE SEMI-LAGRANGIAN METHOD AT DWD

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1. INTRODUCTION

1.1 The models of DWD

The presently operational models of DWD are:

GM (Globalmodell)
EM (Europa Modell)
DM (Deutschland Modell)

Only the DM uses the semi-Lagrangian (SL) method. A short report will be given in section 2.

The new model generation, which is now under development will use the SL-method in order to increase the accuracy of the advection treatment and, if possible, to improve the efficiency slightly. The new model set will consist of two models:

GM_E (New global model)
LM (Local model)

The new models, in particular the LM, will contain an advanced treatment of the water, including an explicit representation of the ice phase. The advection of such spotty fields requires an accuracy of the advection treatment which is not available with numerical schemes that exist in current operational models. We hope to avoid the problems evident in many of the current implementations of the SL-method when using CFL-numbers > 1.

Mathematical considerations of this topic are presented in section 3. A truly noninterpolating scheme based on the Particle-in-Cell method delivers the accuracy of advection required for LM. This research effort will be reported in section 4. The new global model GM_E uses an almost uniform triangular grid. Research relating to the development of such grids is reported in section 5.

1.2 Definitions

In the past, the interest concerning the meteorological applications has been motivated very much by expectations of very efficient schemes, which are possible when rather large timesteps can be used. The results obtained during the last ten years with the SL - method fall considerably short of the original expectations, as with three-dimensional models only moderate increases of the CFL have been achieved. In order to remind of the aims to be followed by SL-research, the following definitions may be useful.

DEFINITION 1 (Efficiency):

An SL scheme is efficient if CFL-numbers much greater than 1 can be used.

DEFINITION 2 (Accuracy of advection)

A numerical scheme is defined to be of suitable accuracy, if it is exact for homogeneous advection and does not create artificial maxima or minima by the numerical process.

The SL application used in DM falls short of these definitions, as CFL is limited to the order of 3, and for homogeneous advection it produces smoothing and oscillations. The noninterpolating scheme (Particle in Cell) is of suitable accuracy in the sense of definition 2. In order to facilitate the discussion in section 3, the following definition of a deforming transformation is given.

DEFINITION 3 (Deforming transformation)

A coordinate transformation is called deforming if a smooth velocity field in coordinates x, y, z is no longer smooth in the transformed coordinates $x1$, $y1$, $z1$.

It is clear that an SL scheme with large CFL can work only in one of the coordinates x, y, z or $x1$, $y1$, $z1$ if these coordinates are connected by a deforming transformation. The coordinate where SL with large CFL is not to be applied may be called the deformed coordinate system.

An example of a deforming transformation is the transform to geographic coordinates. In this case it is general practice to consider the geographic coordinates as the deformed coordinates, and compute the trajectories as great circles. In section 3 it will be proposed to perform the same considerations for coordinate transformations associated with orography. Such transformations are deforming, if the orography is steep. Obviously, very small scale models, like LM, will suffer most from such problems.

The conclusion of section 3 will be that an SL-scheme with CFL >> 1 will work in the σ-η-system only if every plot of velocities on a p-surface shows non-smooth velocity fields. Each time a plot of velocities in the p-system is smooth, we have a situation where σ-η-SL-schemes with CFL >> 1 are not possible.

2. The SL version of the LM

Since our model is limited to a timestep of 5 minutes by the physics package, we have never had the ambition to achieve a very substantial saving of computer time by employing the SL-method. With dx=14 km the Eulerian version uses a timestep of 90 sec. The SI version is able to use a timestep of 4 min. It is a standard interpolating SL scheme with third order interpolation of the fields, bilinear interpolation for the calculation of the trajectories and three time levels. Some of the trajectory averaging procedures, which have been proposed in the literature, have not been implemented for simplicity. The SL version results in a saving of computer time by a factor of 1.6, which has been necessary for the operational implementation.

As the Eulerian version of DM uses second order differences, an improvement of the accuracy of advection is expected in the SL version. Fig. 1a,b gives forecasts of the 10 m winds with the Eulerian and the SL versions of DM. While in most places the forecasts are rather similar, the SL version produces a southerly turn of the wind in the upper Rhine valley, which is typical for this area.

The SL model has some problems near the stratosphere in areas where the CFL-number is larger than one. Fig. 2a,b shows the forecasts of high clouds for the two model versions. The SL version has an unrealistic small scale structure of the clouds, which disappears when the timestep is reduced to that of the Eulerian version. For operational applications this problem has been overcome by vertical nesting, but it points to a serious problem of the SL implementation which has to be solved in the LM. Many other models also appear to have problems with the SL method in the stratosphere.

3. Theoretical considerations (smooth SL schemes)

According to definition 3 of the introduction the transformation of x, y, z to x, y, σ is a deforming transformation, whereas that of x, y, z to x, y, p may be considered as smooth. Most applications of the SL-method have been performed in the σ-system, with the implicit assumption that the velocity field is smooth in this coordinate. This would necessarily imply non-smooth velocities in the z- or p-systems, somewhat in contradiction to model results, which give smooth velocity fields in the p-system on occasions. It is not quite clear, whether a coordinate system with smooth velocity fields exist, in particular in very small scale models like LM, with mountain induced gravitational waves. It seems, however, that the x, y, z or x, y, p schemes are better suited than other systems for the application of SL schemes. It is not necessary that the whole model or the grid be formulated in this coordinate, however, for the advection to be approximated by smooth trajectories.

All variations of SL-schemes, including the noninterpolating (Particle-in-Cell), can be treated in this way. To be specific, we will describe here the case of a three time level interpolating method where vertical advection is treated implicitly Eulerian.

As a specific example, the undeformed scheme is described here for the case of a three time level interpolating scheme with linear interpolation for the trajectory calculation and third order interpolation for the fields. A scheme treating the horizontal coordinates by the SL-method and the vertical by finite differences as assumed with σ- or η-coordinates to define the model levels. We think of a limited area model with tangential coordinates, in order that horizontal coordinate deformations are not present. Generalization to global models is easy, since the principles proposed here are common practice in respect of the horizontal coordinates in order to avoid the pole problem. Any other SL scheme can be treated in a similar way, including noninterpolating SL (Particle in Cell).

In order to modify the SL scheme to undeformed SL, three steps have to be performed:

- the vertical advection has to be done in the p-system, using $\dot{p}\dfrac{\partial f}{\partial p}$ as vertical advection term.

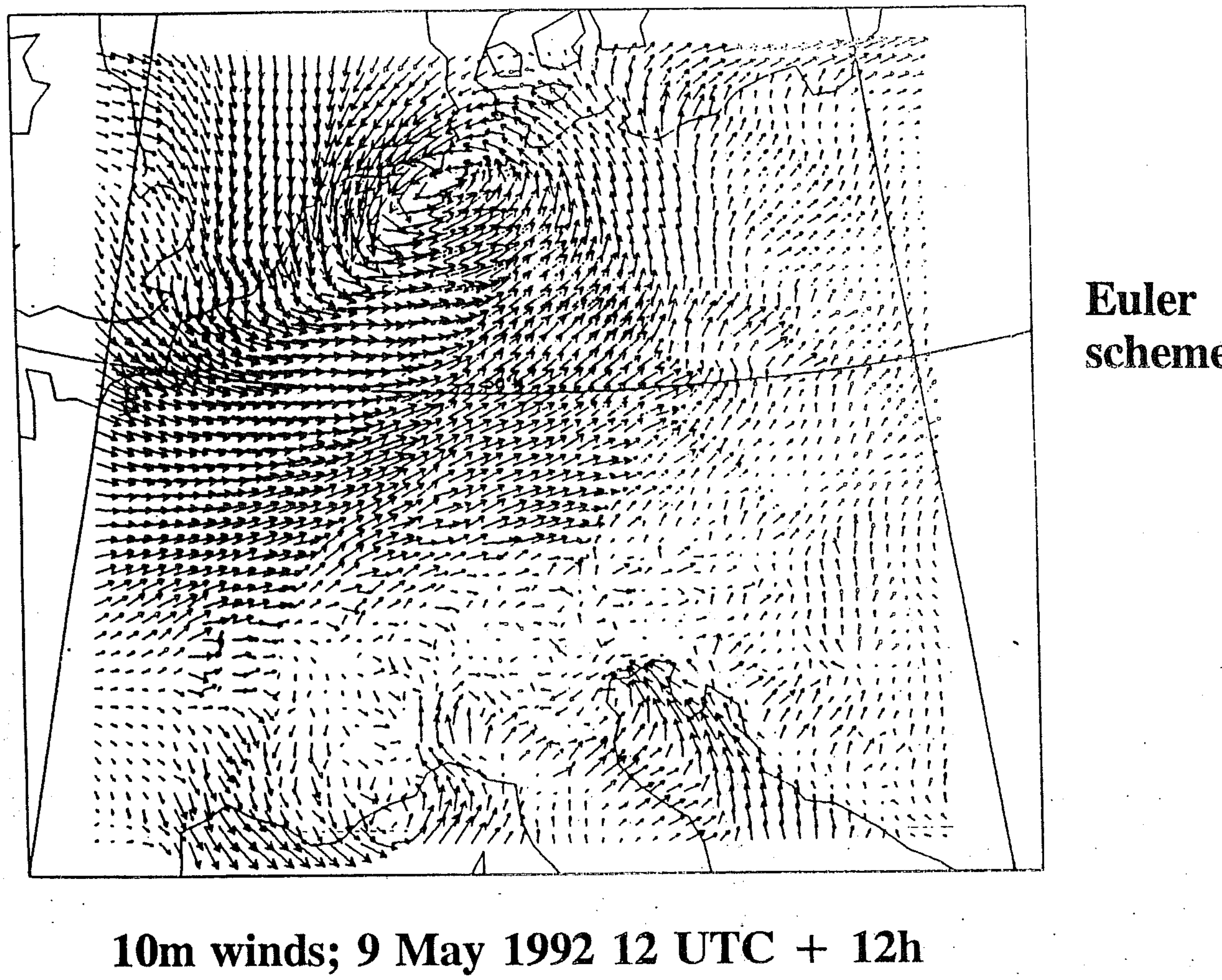
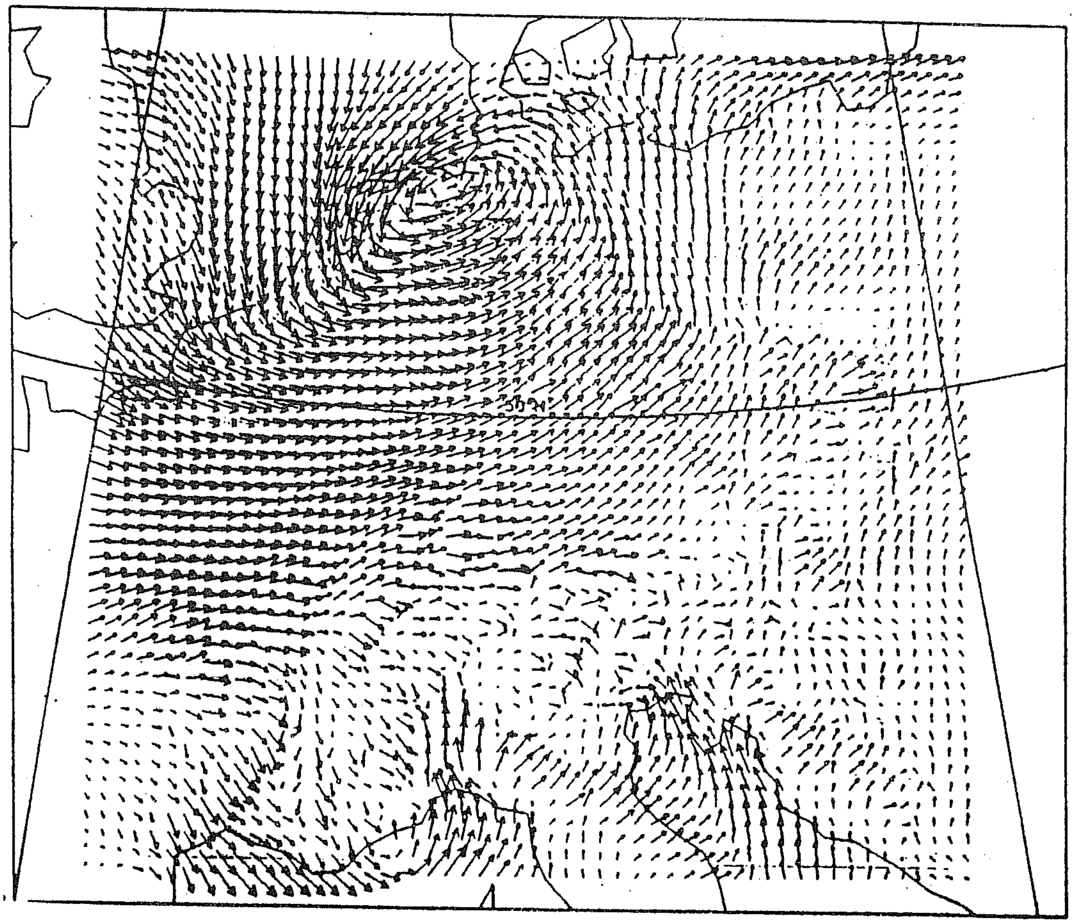


Fig. 1a:
Forecast of the 10m-wind by the Eulerian version of DM.



SL scheme

Fig. 1b: As fig. 1a, for the SL-version of DM.

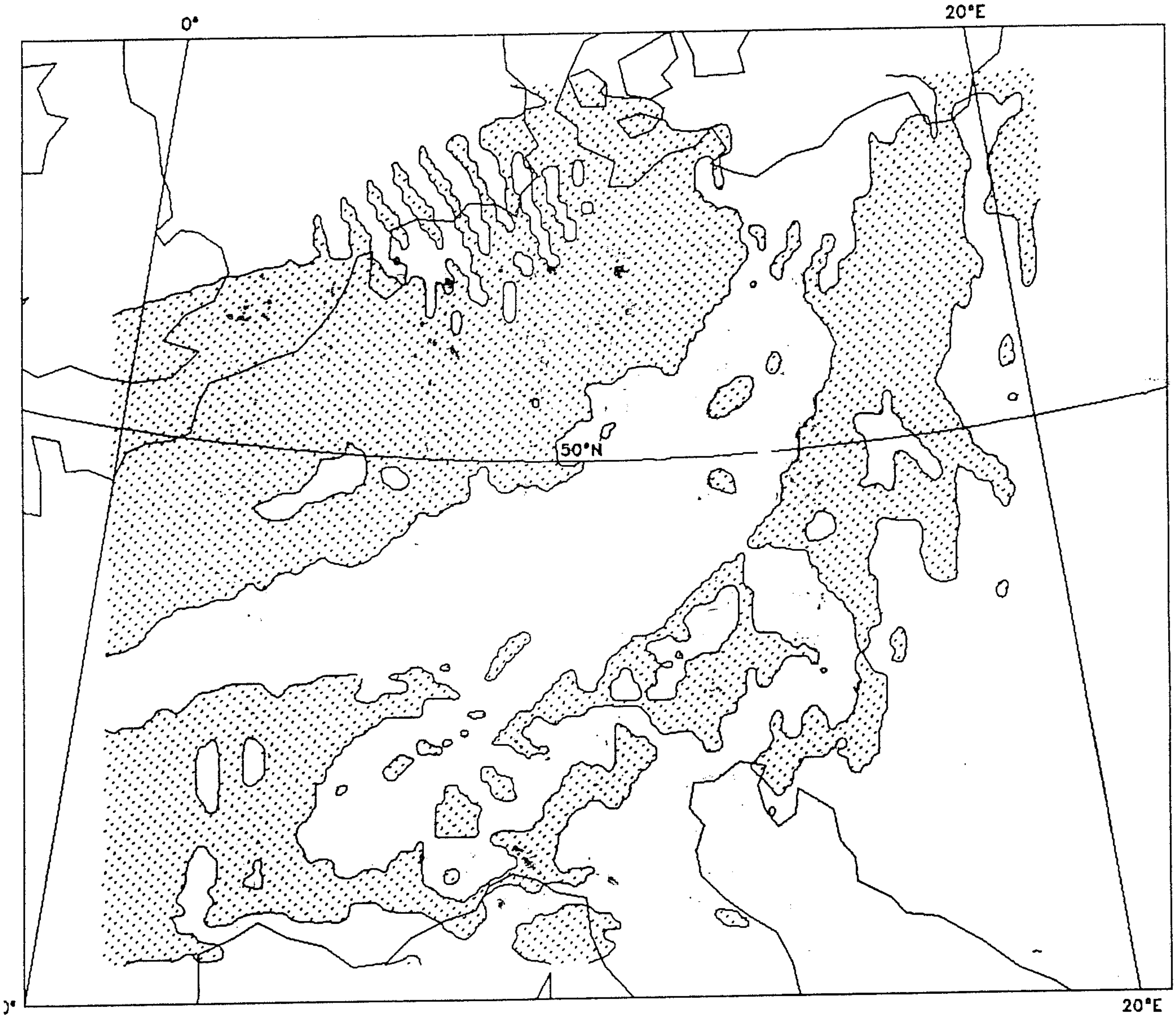


Fig. 2a: Forecast of high clouds with Eulerian version of DM.

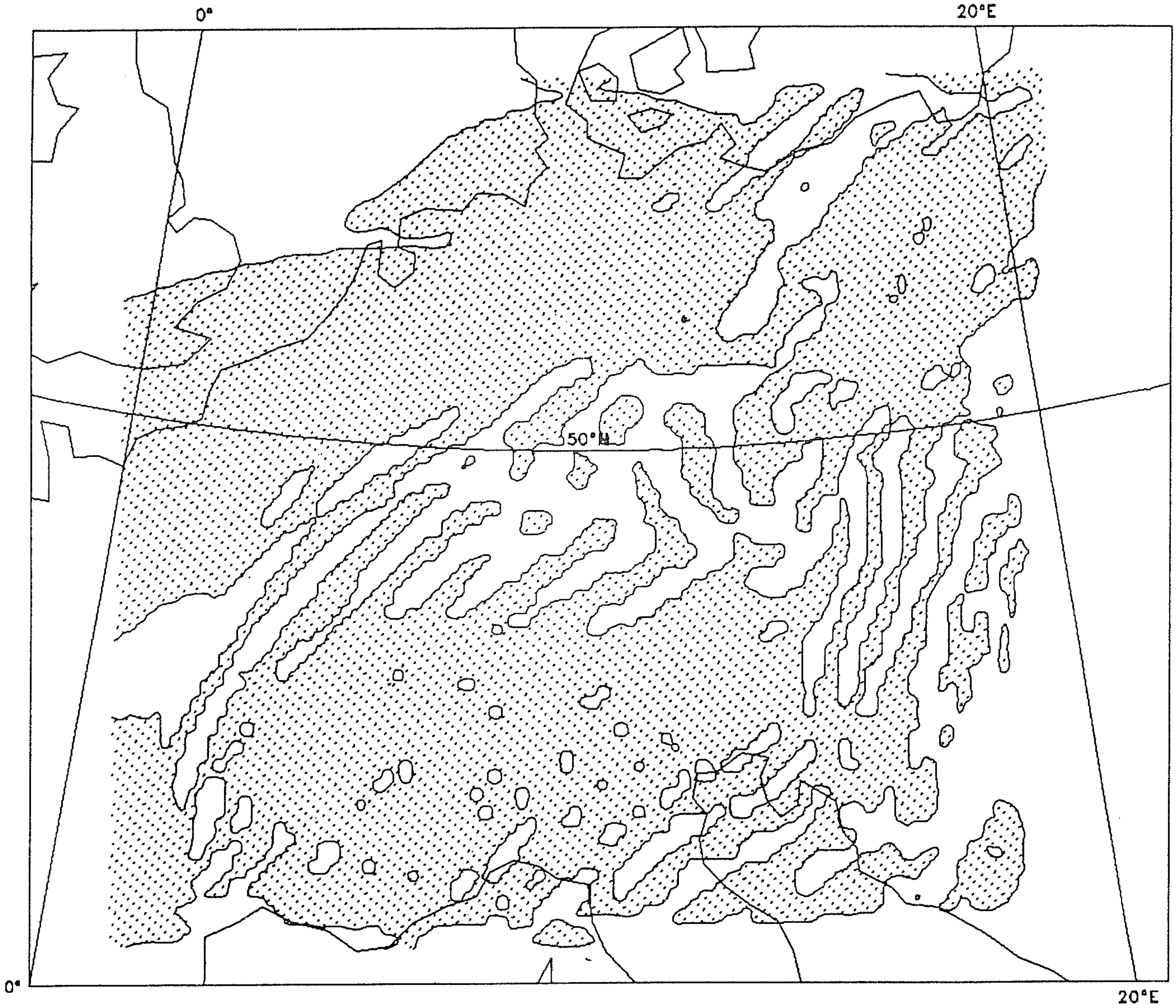


Fig. 2b: as fig. 2a, for the SL-version.

- The trajectory calculation is done by iteration, as usual. With k being the iteration index, and n the time level, we obtain for the first step:

$$\underline{\alpha}^n_k\ (x_i,y_j) = -2dt\ \underline{U}^n\ (x_i,y_j)$$

$\underline{\alpha}$ is the trajectory at t_{n-i}, $\underline{U}$ is the horizontal velocity vector and x, y are the horizontal coordinates. The other iteration step is performed as follows:

$$\underline{\alpha}^n_{k+1}\ (x_i,y_j) = -2dt\, I(\underline{U}^n)((x_i,y_j) - \frac{1}{2}\underline{\alpha})$$

I is the linear interpolation operator.

- The fields f^* at time n-1 at the end of the trajectory for a field f are computed as follows:

$$f^*_{i,j} = J\ (f^{n-1})\ ((x_i,y_j) - \underline{\alpha})$$

J is the high order interpolation operator. Note that J will also interpolate in the vertical, even though only horizontal advection is treated by the SL scheme. In the special case of linear interpolation for the vertical, it is possible that quite different levels are used for the interpolation than that of the starting point.

4. Noninterpolating SL schemes (particle in cell)

The schemes described in this section obey the definition of schemes of suitable accuracy for advection as given in section 1.

The scheme is here described for the special case of a rectangular mesh. The results, however, will be presented for the simpler case of a rectangular mesh. Furthermore, the transformation method between Eulerian and Lagrangian mesh is described here, which so far has not been used in the practical applications. The practical calculations have so far been done using linear interpolation to go forward and backward between the grids. The interpolation may be expected to be responsible for some difficulties of the method when applied to all fields, that have been reported by Steppeler and Orszag (1995). The introduction of the transformation method, as described in the following, can be expected to resolve such problems. It is also much more suited for operational use, since the field representations in the Eulerian and the Lagrangian grids are equivalent. Numerical work to prove the suitability of the method when applied to all fields needs still to be done. Up to this point the noninterpolating SL method may be applied to density fields and potential temperature alone. This is planned for the model LM. The Danish Meteorological Office seems to work on similar methods, and has also reached the conclusion that at the present point of development the method should be applied only to density fields. However, no definitive information on the work in Denmark has so far reached the author.

Let the triangles be indexed by i. In practice double indices can be used, as common in atmospheric models (Steppeler and Prohl, 1995). The method is described for the two dimensional case. The centre of each triangle is $(x_i,\ y_i)$. For each triangle also a Lagrangian point (xl_i, yl_i) is defined. It is assumed that each Lagrangian point is within a distance a from the corresponding Eulerian grid point (x_i, y_i). It is essential that a is larger than the diameter of a triangle, for example, double this distance. In this way a certain Lagrangian point may be associated to more than one Eulerian point or there exist more than one indexing of the Lagrangian points. For any field f gridpoint values f_i in the Eulerian grid and fl_i in the Lagrangian grid are assumed. These values are not independent, but obtained by transformations, which will be defined below. Using field values in the Eulerian grid, trajectories can be computed in the usual way. The timestep going from n to n+1 is then defined as follows:

$$(xl_i\ yl_i)^{n+1} = (xl_i\ yl_i)^n + \alpha_i$$

The $(xl_i,yl_i)^{n+1}$ obtained in this way do not necessarily yield a Lagrangian grid, since the requested connection to the Eulerian grid may have been destroyed. A Lagrangian grid at time n+1 is obtained by a reordering or grid smoothing step. In the examples to be reported, the first come, first served principle is used for simplicity. It has the disadvantage that it depends on the indexing of the Lagrangian points. Each Lagrangian point (xl_i, yl_i) is given in the index of that Eulerian point to which it has the smallest distance. In case that fails, the other Eulerian points within the distance a are tried. Points which cannot be indexed by this procedure are not used. If any index is not used, a point is created by interpolation from the Eulerian grid. As described, the procedure is not mass conserving, but this could be changed easily.

An explicit timestep is then performed in the following way:

- the fields are transformed to the Lagrangian grid
- the advection step is done in the Lagrangian grid, including the grid smoothing. The advection tendencies are transformed back to the Eulerian grid.
- the timestep for the non-advection portion of the step is performed in the Eulerian grid.

The transformation between the grids remains to be defined. Let L be any linear interpolation operator from the Eulerian to the Lagrangian grid. The back transformation is then defined to be the inverse transformation. This is obtained by solving the linear set of equations defined by L. This procedure has the advantage that it is mass conserving and does not depend on different indexing of the Lagrangian representations. According to our tests this procedure is numerically not efficient, and numerical difficulties arise in a situation when the determinant of the equation is small. Therefore the following modification of the procedure is proposed, which is numerically very efficient, and avoids problems of ill conditioned matrices.

Let the original linear interpolation for a field f be:

$$fl_i = \lambda_1 f_i + \sum_k \lambda_k f_{ik}$$

where the summation is over neighbouring points of the point I. The procedure is modified to

$$fl_i = \max\ (½,\lambda_1) + \sum_k \lambda_k^* f_{ik}$$

In this way the matrix is diagonally dominant. In the case of a rectangular grid this modification of the interpolation can be defined in such a way that a combination of two fast Gaussian elimination procedures can be used.

As an illustration the Lagrangian grid for a computation of a Rossby wave, which in a barotropic model moves over a mountain, is shown in fig. 3.

Fig. 4a-c shows the results of a computation of a convective cell, where only the potential temperature is computed by the noninterpolating SL-method. Fig. 5a-c gives the corresponding results using centred finite differences. Strong violations of the material conservation of potential temperature show up right in the beginning of the forecast. The comparison to the computation by second order finite differences shows that the material conservation of the potential temperature is much better represented with noninterpolating SL. In the beginning of the calculation artificial maxima and minima are absent, and they occur only at the end of the calculation, when the high temperature layer has a diameter of only one gridlength. The maxima occurring then are caused by the granularity of the discretization. Accurate representation of steep gradients is present up to the smallest scales of the discretization.

5. Triangular meshes

Triangular meshes as proposed by Baumgardner (1985) cover the sphere rather uniformly. This grid is obtained by starting from a isocahedron and dividing the sides of the triangles to obtain the refined grid. The horizontal resolution is defined by the parameter *nl*, providing the number of intervals of partition of the original isocahedral triangles. The number of the discretization triangles is then 20**nl***2. In order to facilitate the use of the multigrid method for the solution of the Helmholtz equation for the SL-method, it is convenient to choose *nl* as a power of 2. The better performance of the Baumgardner (1985) method as compared to the work by Williamson (1976) is due to two developments:

- The grid is somewhat more regular.

- The equations are solved in a coordinate system local to the gridpoint, rather than using a global geographic coordinate. Alternatively, a centred earth of a similar undeformed coordinate system can be used.

An example of the triangular grid is shown in fig. 6a. Fig. 6b shows the hexagonal control volumes when using the vertices as gridpoints.

The original work of Baumgardner concerned the flow inside the earth using the finite element method. Applications to the atmosphere have only been done quite recently. For atmospheric modelling simple local approximations seem better suited than the finite element method, which has been used for the inner earth calculations. Two approaches have been tried so far. Baumgardner used a finite difference concept defining the amplitudes at the vertices of the triangles. Steppeler and Prohl (1995) defined the amplitudes of the fields at the centres of the triangles and used the (Eulerian) finite volume approach to obtain local finite differences. There

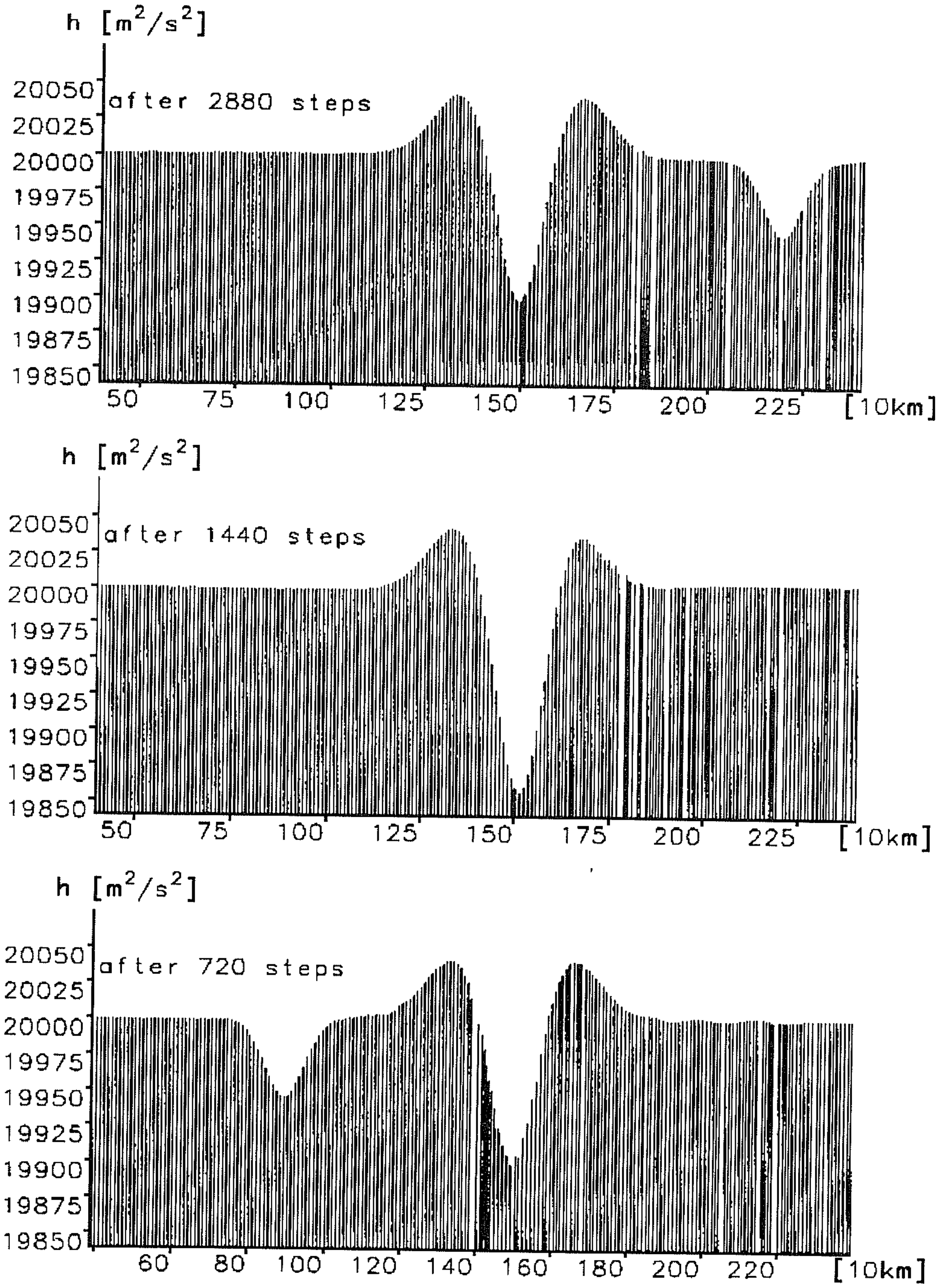


Fig. 3: Forecasts of a one dimensional Rossby wave in a barotropic model using the noninterpolating(particle in cell) SL-method.

a)

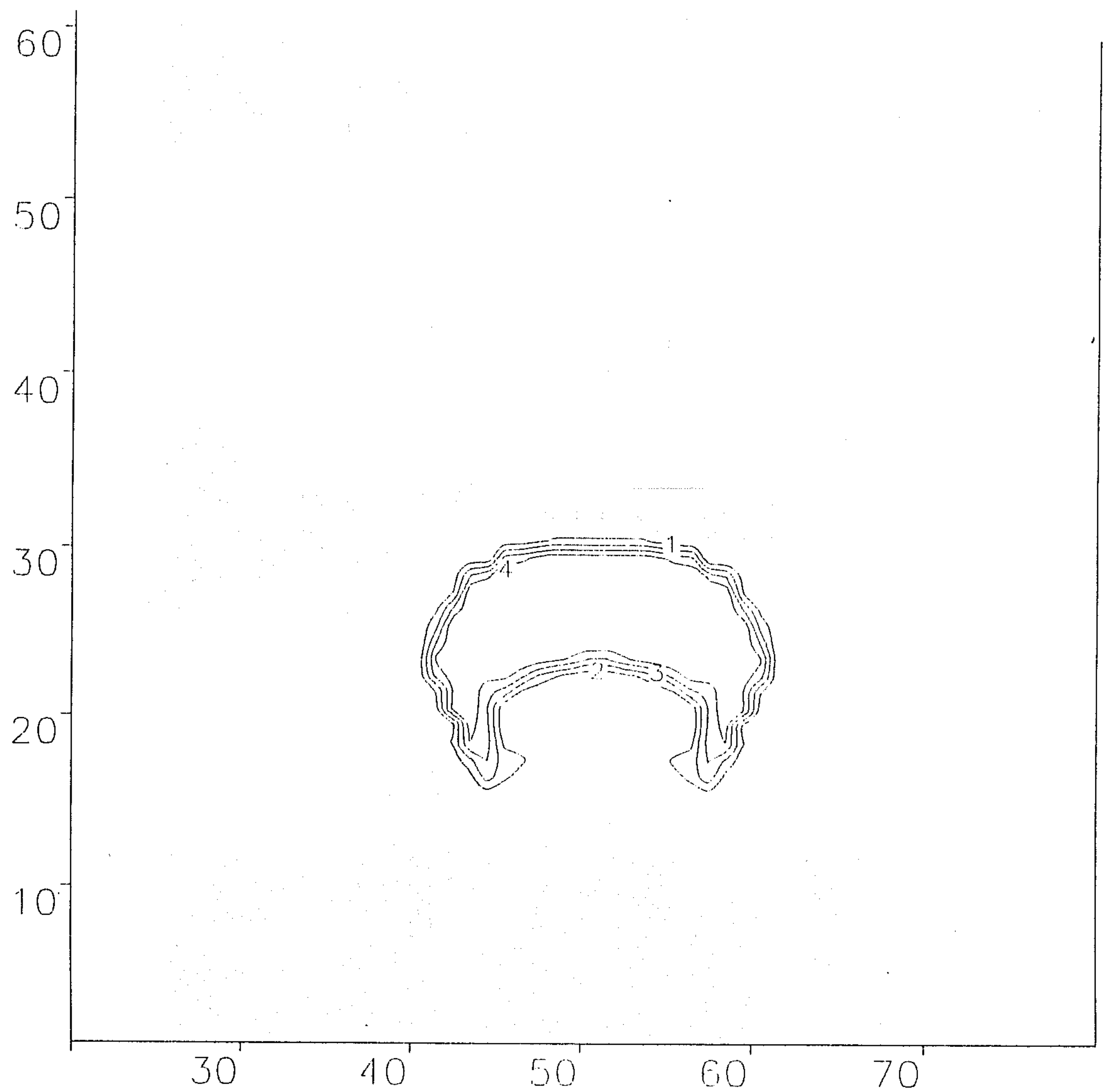


Fig. 4a-c: Prediction of a warm air cell using the noninterpolating SL-method(partice in cell) for the potential temperature

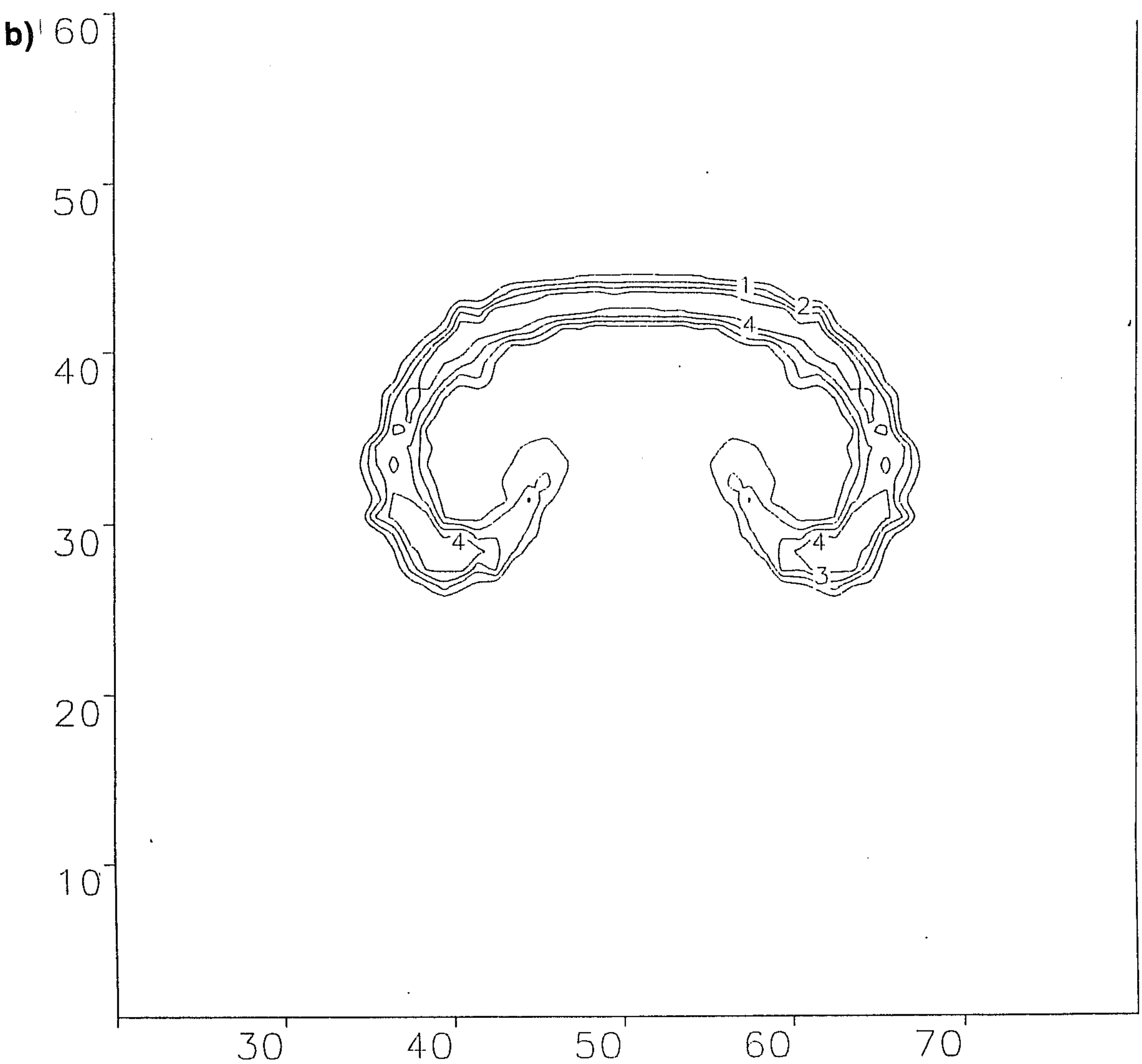


Fig. 4b

c)

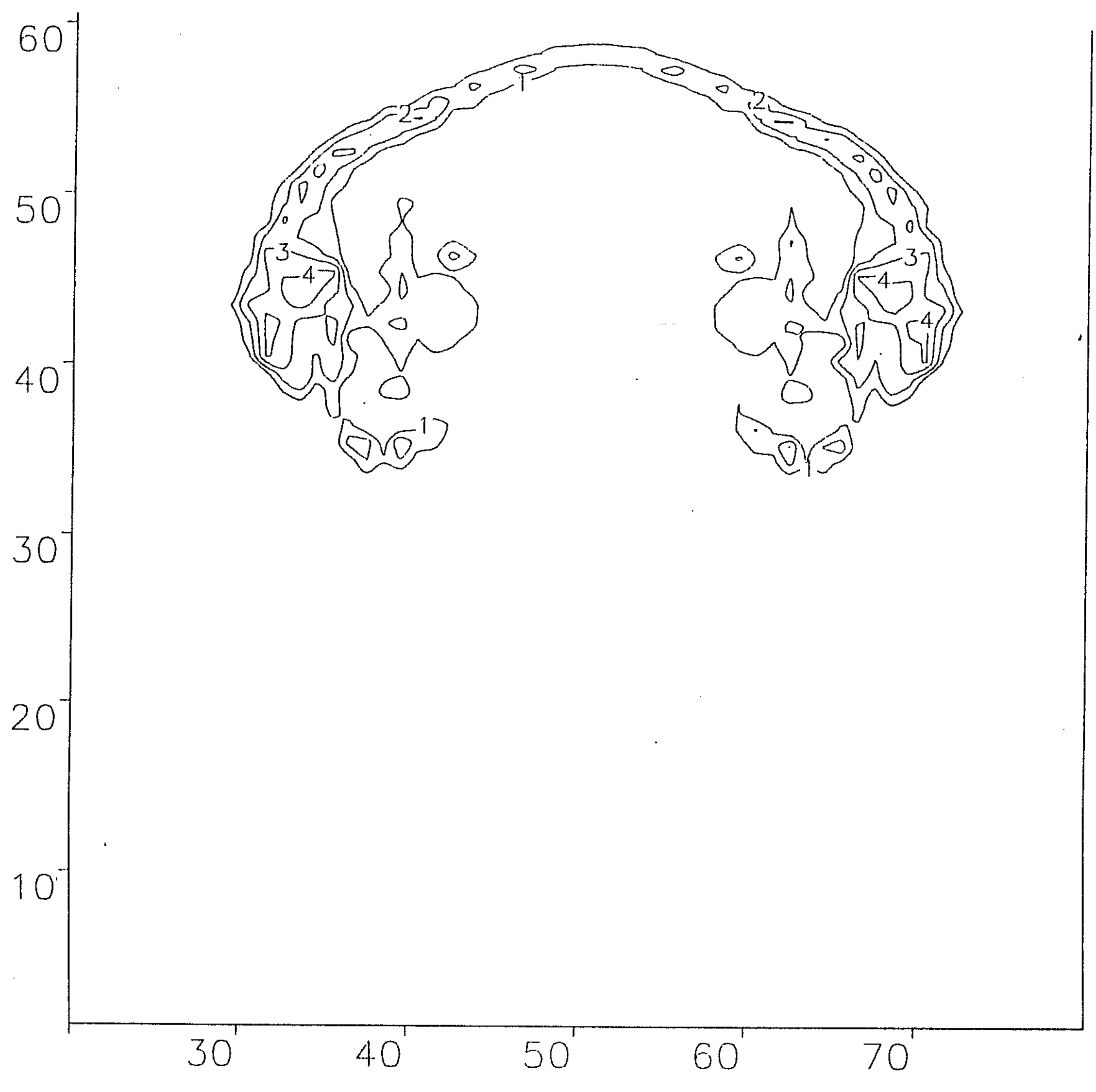


Fig. 4c

a)

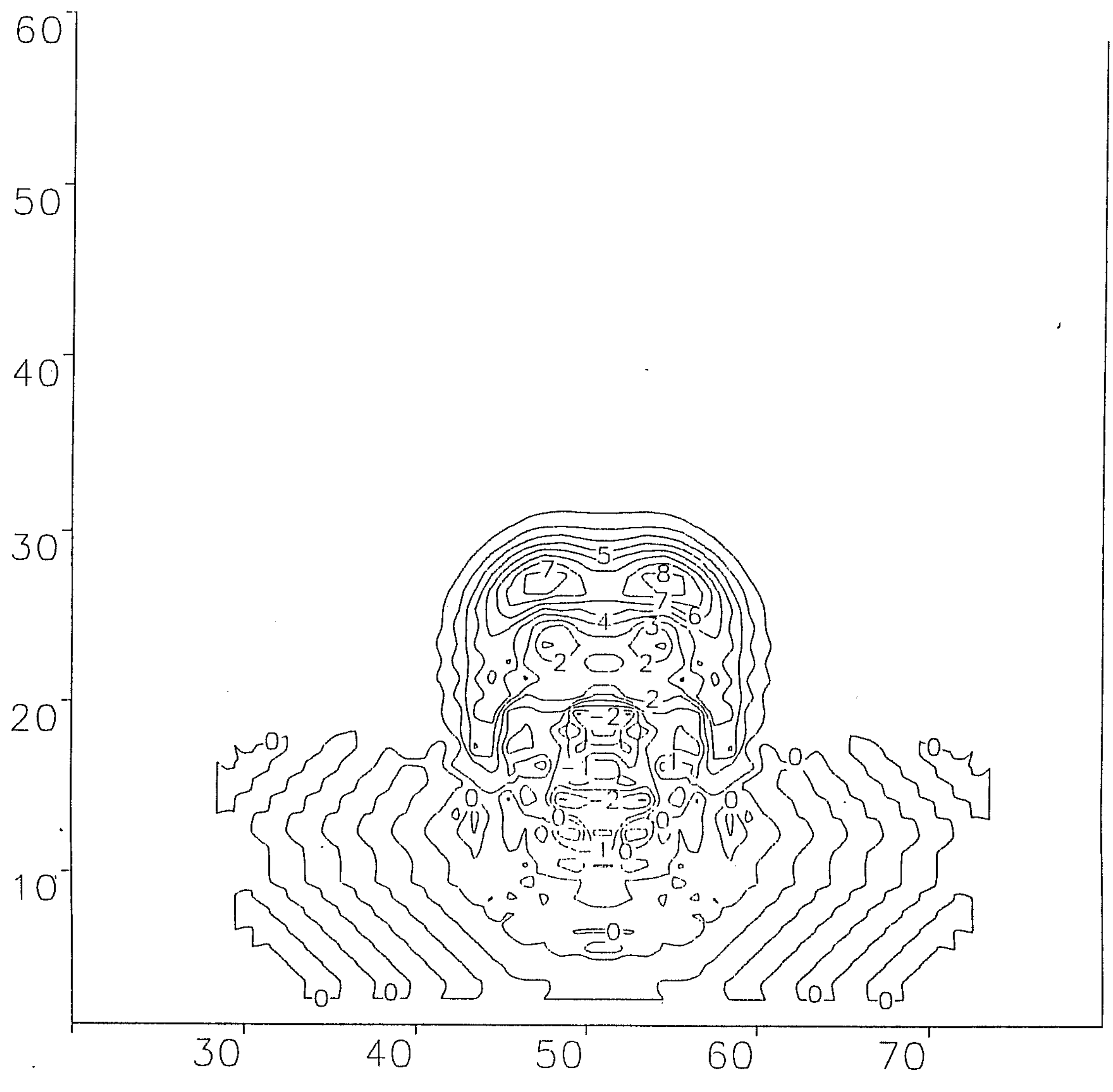


Fig. 5a-c: As fig. 4a-c, for centered finite differences.

b)

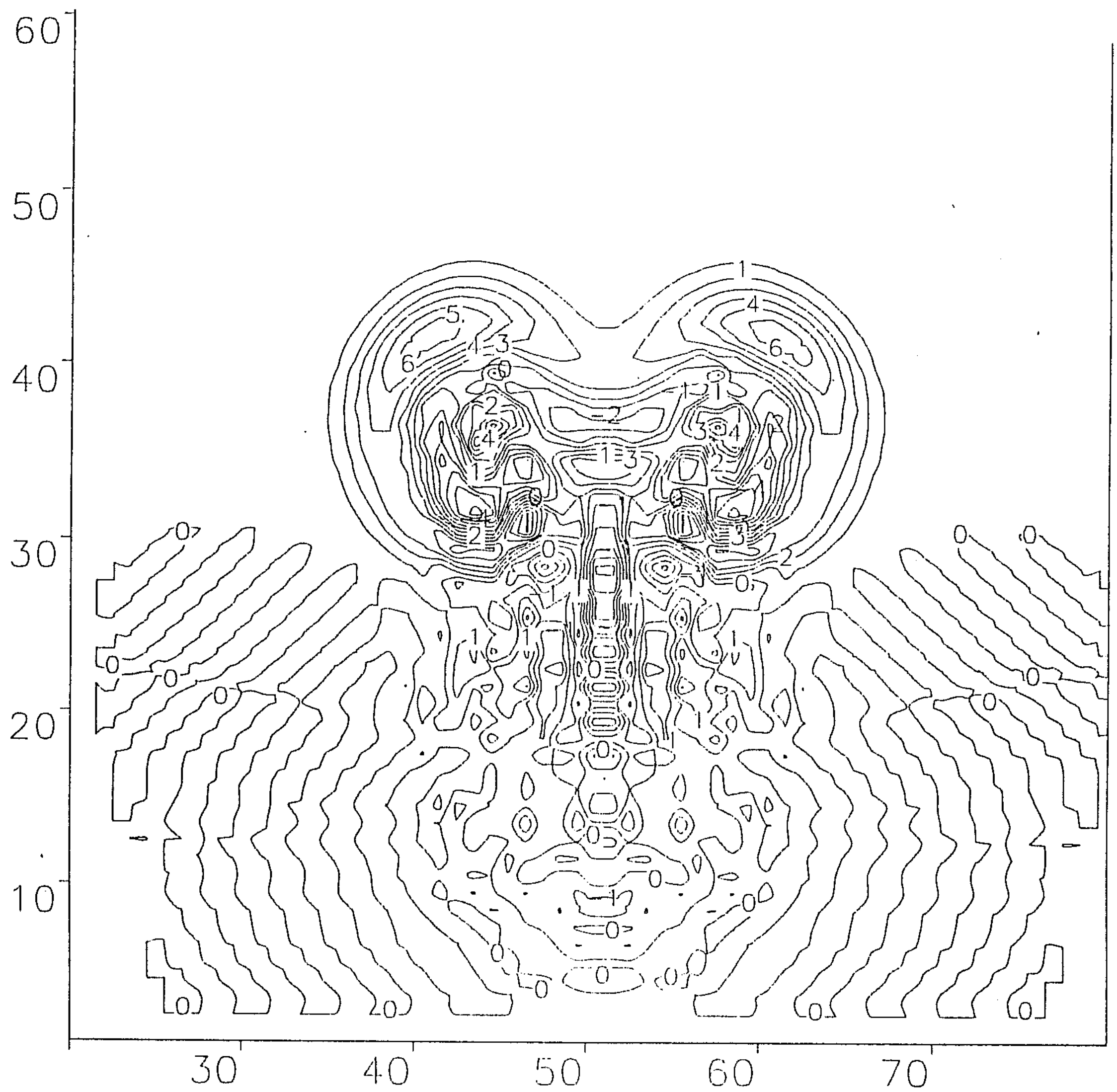


Fig. 5b

c)

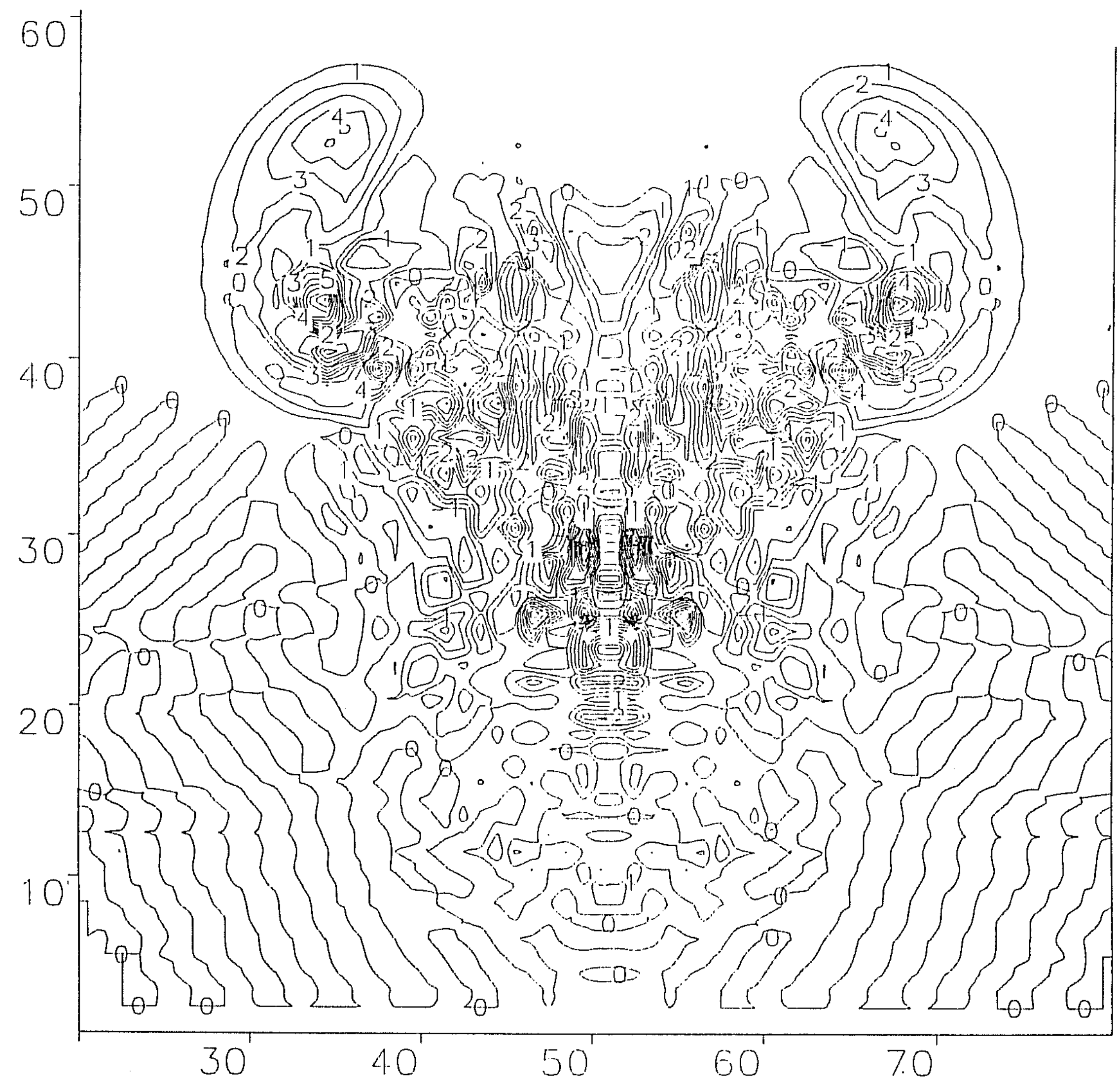


Fig 5c

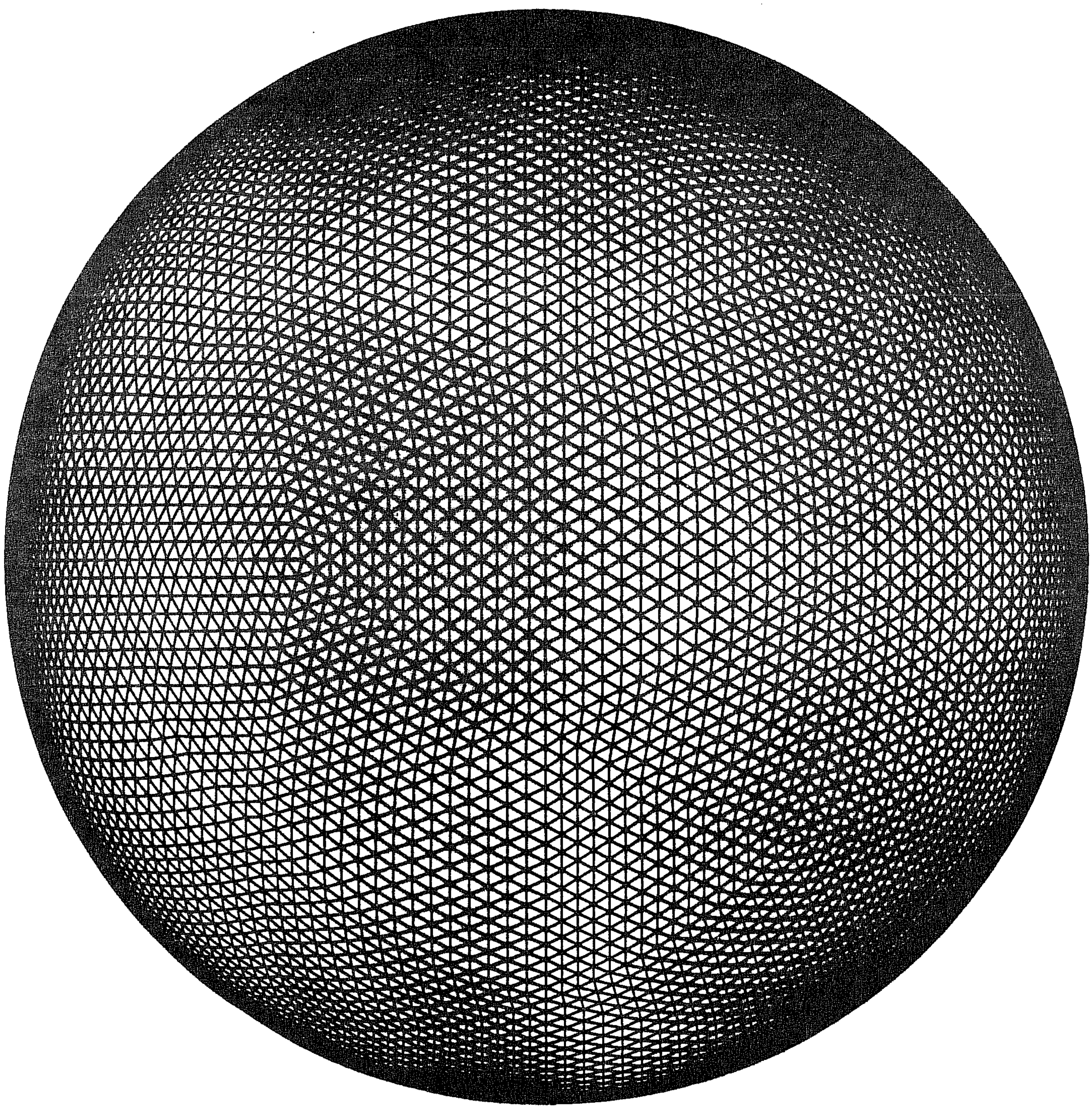


Fig. 6a: The triangular mesh

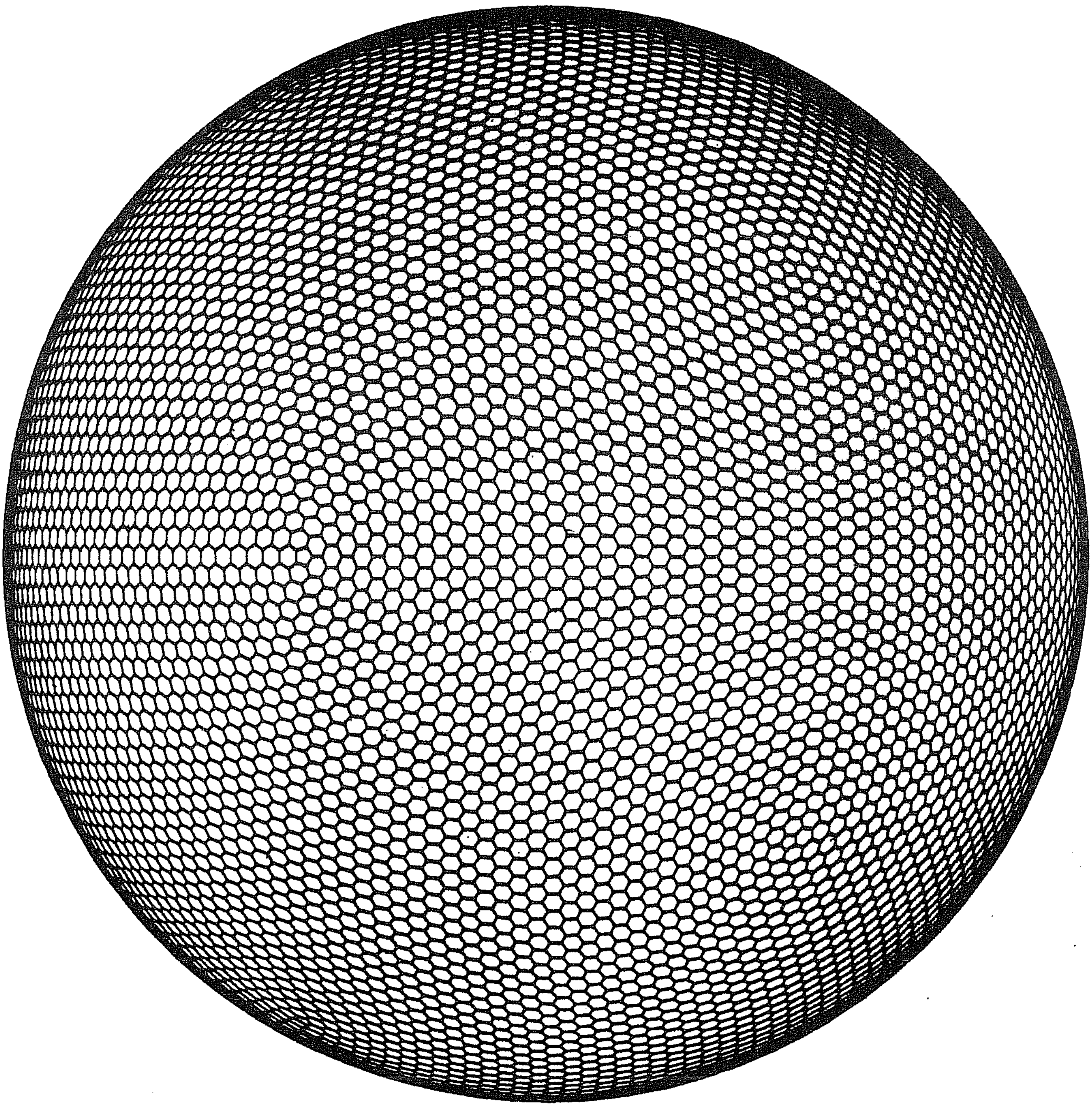


Fig. 6b: The hexagonal control volumes.

are about twice as many triangles than vertices in the grid. Baumgardner used the SL semi-implicit approach with second order interpolation. Both methods have been applied to the barotropic model, and the tests proposed by Williamson (1995) have been used. In all tests the Baumgardner approach performed better than that of Steppeler and Prohl, which could be expected as the latter used second order finite differences, which are not very accurate. As an example which shows that the triangular method does not have a pole problem a Rossby Haurwitz wave producing a cross polar flow is shown in fig. 7. The initial field and a 4-day forecast by the Eulerian finite volume method is shown.

The Eulerian barotropic finite volume method showed the expected reduction of computation time compared with the spectral method. Gridpoint applications can be expected to allow about double the resolution relative to the spectral method for the same computation time. The SL model of Baumgardner so far does not deliver this expected increase of the performance. This is due to the fact that he uses the SL-method to obtain accuracy rather than efficiency. The CFL number of his approach is limited to small values. However, the kind of work necessary to improve this point is rather obvious.

During a recent visit to DWD, Baumgardner developed an adiabatic three dimensional version of his triangular grid model. A preliminary test result is shown in fig. 8.

The SL-scheme used by Baumgardner is a true two time level second order interpolating scheme. The usual approach with two time level schemes to use a third level to interpolate in time is not used. Rather, an iteration procedure is used to compute the fields at half time levels. Barotropic experiments using the SL-scheme with amplitudes at triangle centres have also been performed. Such schemes offer the advantage, that third order interpolation can be performed for the same cost as second order interpolation with the triangle corner grid. However, results sofar have not been favourable.

At DWD the approach to be followed for the new global model GM_E will be decided soon. The triangular mesh model is a strong contender. Arguments in favour of this approach are the following:

- In view of a new generation of fine mesh models there is in many centres a tendency towards grid point models. For example at ECMWF a gridpoint model is approached in very small steps. Presently, the advection is done in finite differences (SL), and the moisture is treated by finite differences at the increased resolution corresponding to T319. Research is going on in order to raise also the resolution of the other fields to T319 by avoiding more of the disadvantages of the spectral method (Hortal, private communication). The moisture field is treated in finite differences on the Kurihara grid, which has been abandoned in the past as it is not very regular near the poles. The triangular mesh offers a considerable improvement in this respect.

- The suitability of the spectral and triangular barotropic models have been compared at NCAR. The triangular model, using only local communication, is considerably better in this respect.

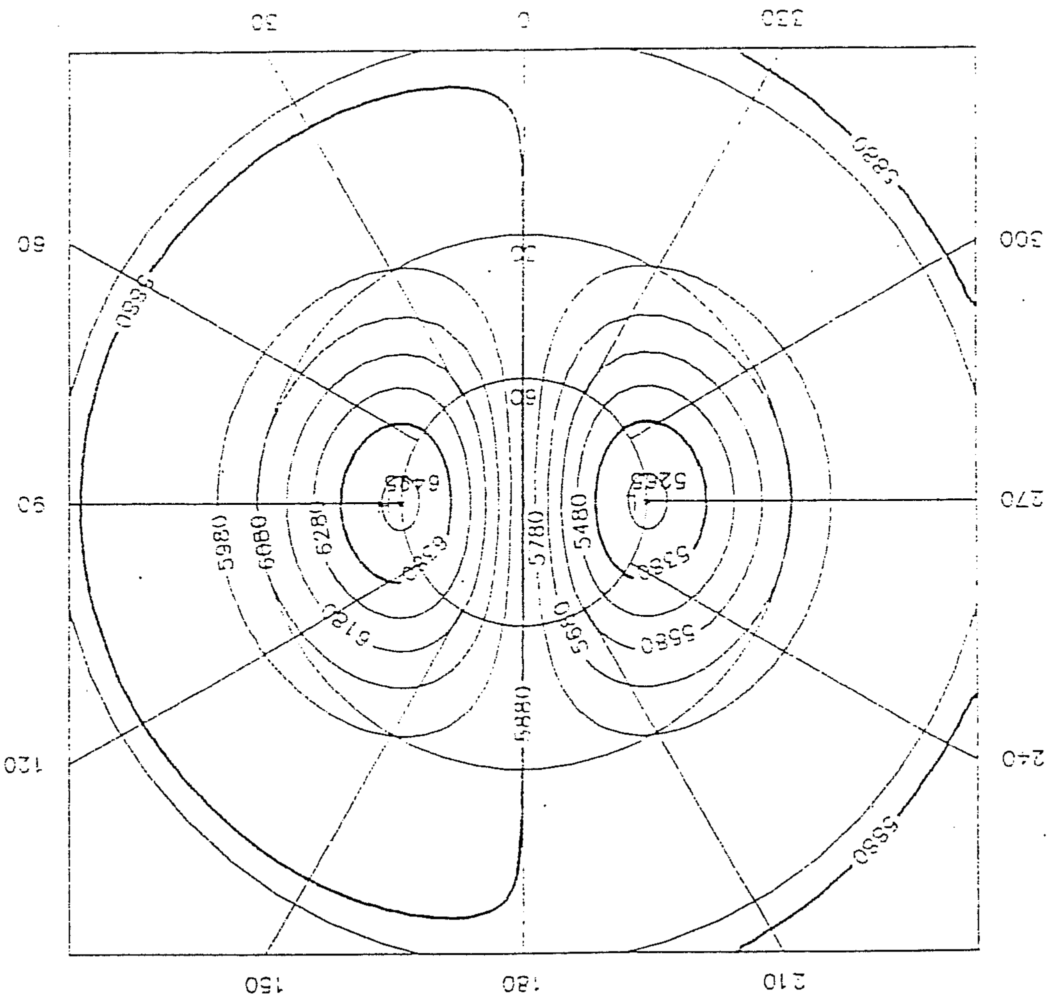


Fig. 7a: Initial values (cross polar flow)

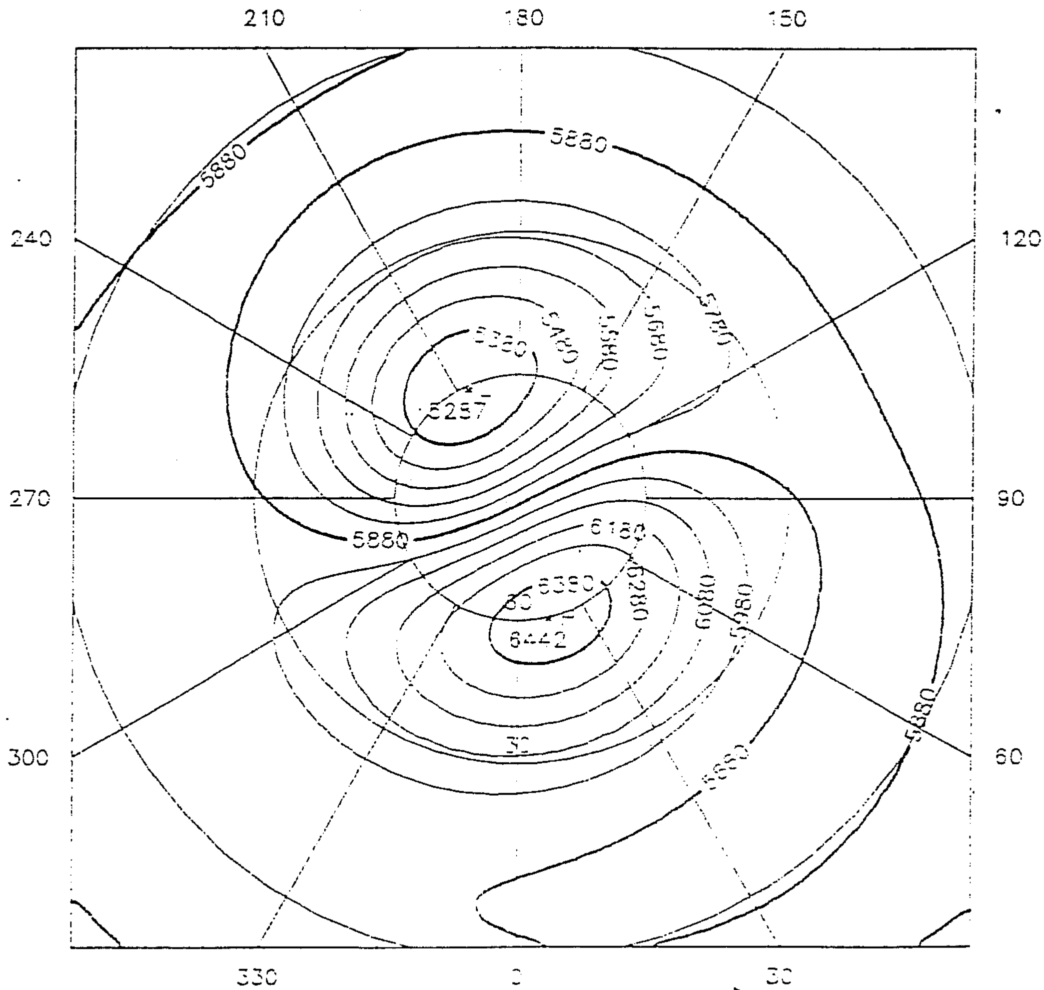


Fig. 7b: Forecast of 2 days.

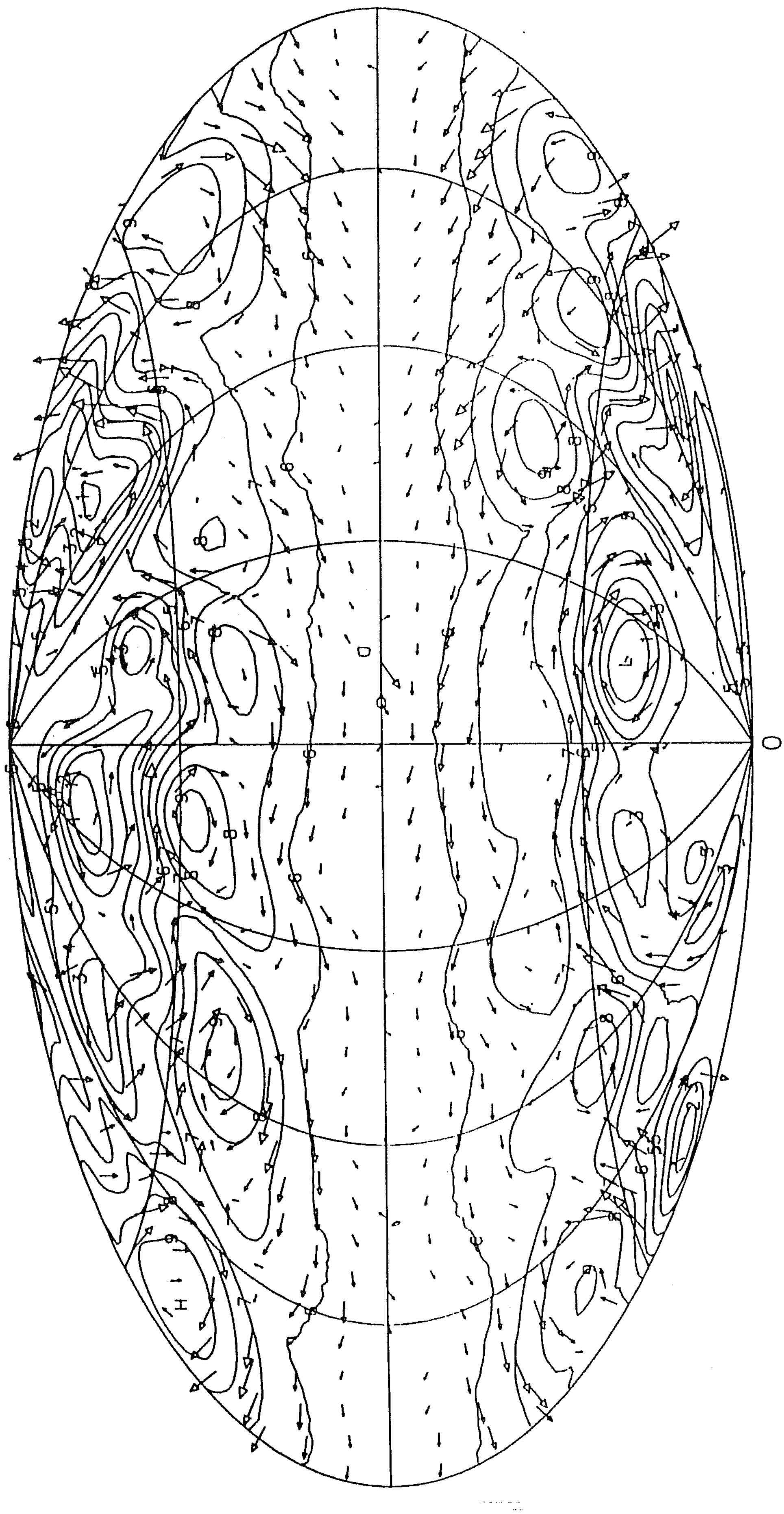


Fig. 8: initial values of surface pressure for the 3-d modelat 11. oct. 95.

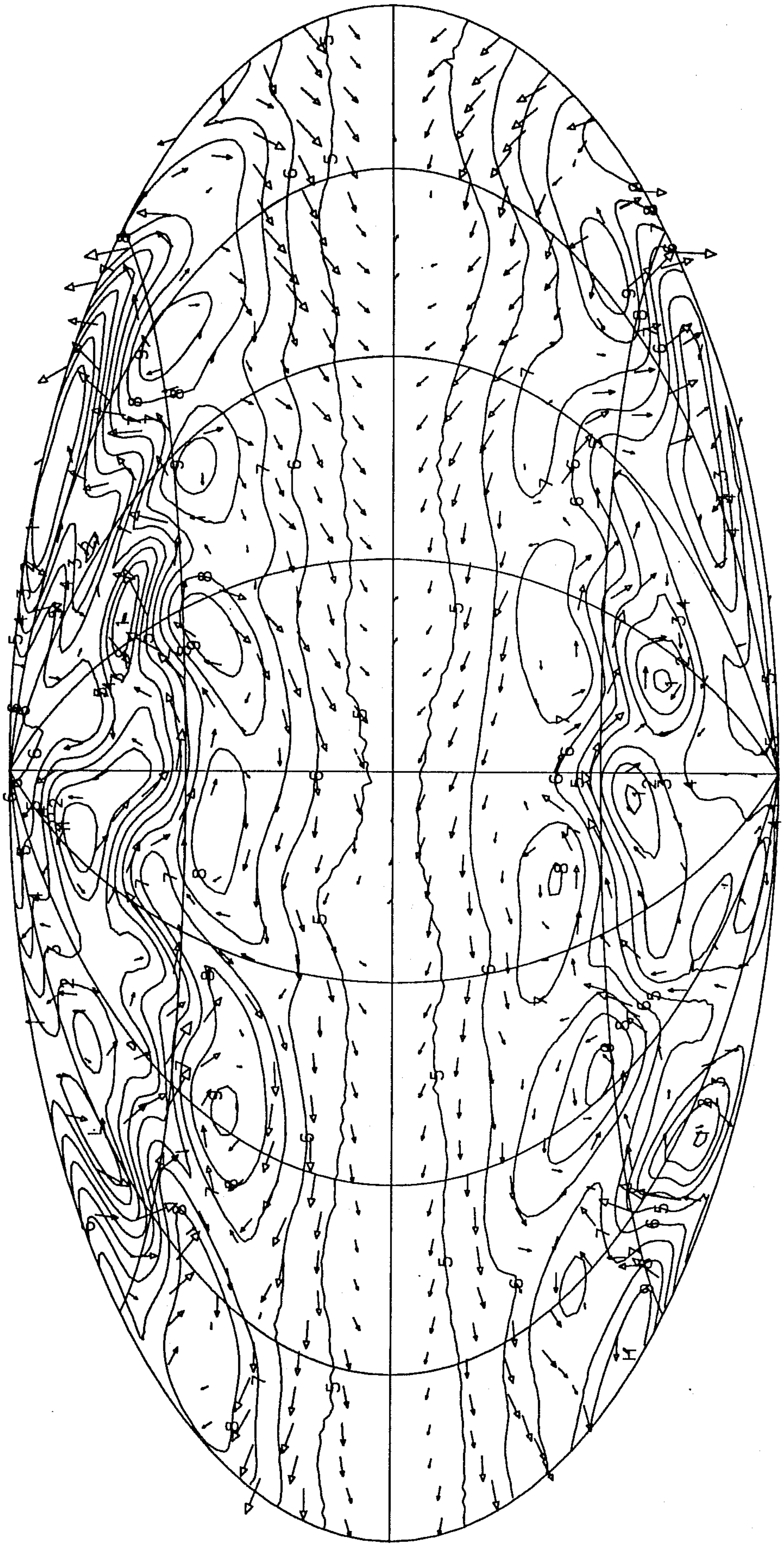


Fig. 8b: Forecast of 4 days using the 3-d triangular mesh model.

- An increased performance as compared to spectral models may be expected, which will allow an increased resolution. This is important since the GM_E will be used to provide boundary values for the very fine mesh model LM.

- The triangular approach is rather simple, and according to experience at DWD model development times have been rather short.

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