VERTICAL DISCRETIZATION PROBLEMS FOR VERY HIGH-RESOLUTION LAM; ARPEGE/ALADIN AS EXAMPLE

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SUMMARY: This lecture is dedicated to new developments in the dynamics of NWP models linked to their application at very high-resolution scales and discusses namely problems regarding the vertical discretization schemes. The ARPEGE/ALADIN model is used here to demonstrate concrete examples of some of these type of new developments, such as the introduction of the elastic dynamics, problems of the proper formulation of the bottom boundary condition or the problem of reflection of gravity waves at the top of the model.

1 INTRODUCTION

The vertical discretization schemes do not seem to evolve that much over the last years: most of the NWP models employ schemes developed in the beginning of the eighties for some type of hybrid vertical coordinate; since then we have not witnessed so many novelties in this field, at least not for hydrostatic primitive equation models. In fact, modifications and new developments of the vertical schemes are forced by changes in the dynamics when going to higher spatial (even if rather horizontal) resolution. Some of these recent developments are presented in this lecture, which is divided into four sections: the first one is very brief and it recalls some basic facts about the ALADIN model, which is used further on for demonstration examples; the second one describes vertical scheme developments induced by the introduction of the elastic dynamics; the third and fourth ones deal with such developments caused by implementation of other changes in the dynamics: radiative upper boundary condition and release of thin layer hypothesis.

2 ALADIN: OPERATIONAL MODEL AND RESEARCH TOOL FOR MESO-SCALE

ALADIN has been developed within an international research and development project to complete the family of IFS/ARPEGE global NWP tools by a limited area version. The philosophy of development has been rather simple: to keep as much as possible from the global model ARPEGE (*Courtier et al.*, 1991), to develop only the missing parts (e.g. coupling scheme at lateral boundaries) and to make adjustments to ALADIN's specific "plane" geometry whenever needed. This strategy minimised the investments, but it encompassed some basic choices: for instance ALADIN is a spectral model like IFS/ARPEGE, though the choice of a spectral method for a limited area model is not obvious, given the difficulty of the treatment of lateral boundary conditions. From the list of known methods to cope with the above mentioned problem:

- perturbation method (*Hoyer*, 1987)
- extension of base of functions (*Tatsumi*, 1986)
- biperiodisation method (*Machenhauer and Haugen*, 1987)

the method of biperiodic extension of fields has been chosen. The limited area domain of the model is completed in both x, y directions by an artificial zone (so-called extension zone), where the fields are extended to fulfil the biperiodic conditions, required by Fourier transforms, as shown on Figure (1). As usual, the spectral representation of variables is useful for application of a semi-implicit marching scheme and for computation of horizontal derivatives.

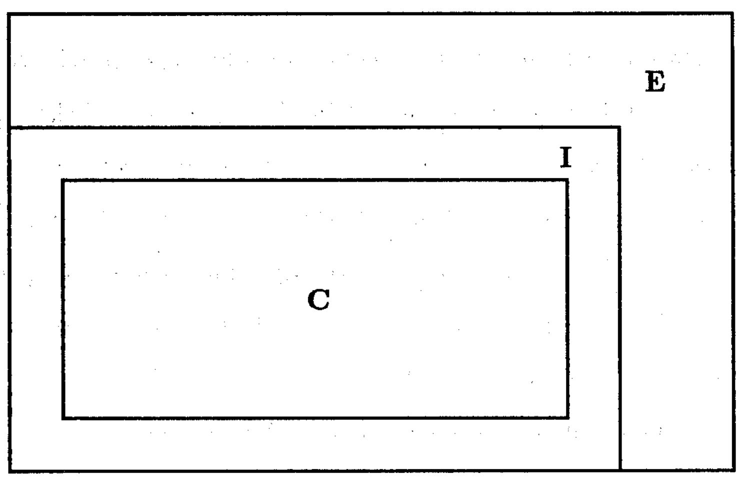

Fig. 1: Domain of ALADIN: C - Central zone, I - Intermediate zone of coupling/relaxation, E - Extension zone.

In ALADIN, this biperiodic extension is applied only for the initial and for the refreshment lateral boundary conditions. Like that, the solution over the extension zone is considered in the same way as a large scale forcing.

As mentioned above, the differences between the limited area model ALADIN and the global model ARPEGE are strictly kept to the following minimum: i) in ALADIN a plane horizontal geometry is used and not the spherical one; ii) double-Fourier series together with an elliptical truncation of spectra are used in ALADIN while spherical harmonics with a triangular truncation of spectra are used in ARPEGE; iii) there is a coupling/relaxation procedure in the limited area model.

On the other hand the following important parts are identical in both models: i) the governing equations of both models employ the thin layer hypothesis; ii) vertical discretization scheme; iii) basic choice of parameterisations for the physics; iv) interpolation operators used in pre/post-processing and in semi-Lagrangian computations.

Currently ALADIN's configurations based on hydrostatic primitive equations are exploited operationally, either in data assimilation or in dynamical adaptation mode. In research mode, other configurations of ALADIN are developed and tested, namely in very high-resolution simulations, using the elastic equations, for example.

3 ELASTIC DYNAMICS

The goal of this section is not to describe entirely the elastic dynamics implemented in ALADIN as extension to the hydrostatic primitive equation set (*Bubnová et al.*, 1995), but, in a less specific way, to see the link between this development and the nature of the vertical coordinate as well as to assess the associated problems in the vertical discretization of some terms. The aim is therefore to show, how the choice of prognostic variables and of the vertical grid staggering is important for the formulation of top and bottom boundary conditions and for introduction of the semi-implicit scheme.

3.1 Vertical Coordinates

In order to introduce relatively easily the orography into atmospheric models, hybrid-types of vertical coordinates are employed nearly everywhere nowadays. The definition of hybrid coordinates is based, in general, on some monotonous function of altitude with a modification matching the orography surface near the bottom. The hybrid co-ordinate may be derived from pressure, height or potential temperature, but since the isentropic surfaces may have a very complicated shape near the surface, we see in practice the two first categories: pressure- or height-based vertical hybrid coordinates.

While most of NWP hydrostatic primitive equation models use a pressure-type vertical coordinate, non-hydrostatic models are in general built with a height-type coordinate (in principle pressure ceases to be a monotonous function of height when the hydrostatic hypothesis is relaxed). The choice of a height-based vertical coordinate is done quite often when a non-hydrostatic model is formulated from scratch (the coordinate of *Gal-Chen and Somerville* (1975) is the usual choice), while the situation is far less obvious for already existing, mostly very complex NWP systems, using pressure-type of vertical coordinates. In the case one would wish to introduce a non-hydrostatic option into the dynamical kernel of such a system, the change of the vertical coordinate inside all the system would represent a considerable development. This is quite a strong motivation for searching the possibilities how to develop the non-hydrostatic dynamics with the use of pressure-based vertical coordinates.

A pioneering work in this direction was done by *Miller and White* (1984) when a set of quasi-non-hydrostatic equations was derived in p co-ordinate, quite close to the classical anelastic equations. A similar approach was adopted recently by *Männik et al.* (1998) resulting in an anelastic set of equations in pressure coordinate, being now under tests in the framework of the HIRLAM model. The fully-compressible (elastic) equations cast in pressure-type of coordinate have been under development as well; most of such systems are based on the so-called "hydrostatic pressure" vertical coordinate, as suggested by *Laprise* (1992). Hydrostatic pressure is a sort of a mass-type variable, reflecting the weight of a column of air above a given point and becoming true pressure if the atmosphere is in hydrostatic equilibrium:

$$\frac{\partial \pi}{\partial \phi} = -\rho g \tag{1}$$

where π denotes the hydrostatic pressure, which is now a monotonous function of height. A terrain-following hydrostatic pressure coordinate is used in ARPEGE/ALADIN to develop an elastic version of the dynamics (*Bubnová et al.*, 1995) and another application of Laprise's coordinate may be found in the Canadian GEM model (*Côté et al.*, 1998). A bit different is the approach of *Dudhia* (1993) used in the Penn-State NCAR Mesoscale Model, where the employed terrain-following pressure coordinate is linked to height by a relation constant in time (there is no temporal variation of the coordinate "density") and thus is closer to a z-type of coordinate.

Using the example of the elastic dynamics developed in ARPEGE/ALADIN, we may examine what are the necessary changes to be done to go from the hydrostatic primitive equations (HPE) to the elastic ones. First, the *Simmons and Burridge* (1981) -SB81-, vertical coordinate used in

ARPEGE/ALADIN is redefined by employing hydrostatic pressure, instead of true pressure:

$$\pi = A(\eta) + B(\eta)\pi_s \tag{2}$$

where π_s denotes surface hydrostatic pressure. Further, there are two additional prognostic equations in the ARPEGE/ALADIN elastic equation set compared to HPE: i) the equation of vertical acceleration completing the existing horizontal momentum equations; ii) the equation for true pressure. The remaining equations are slightly modified with respect to their HPE form in the following manner:

- The horizontal momentum equations "feel" non-hydrostatic geopotential fluctuations, via the horizontal pressure gradient term.
- The forcing term in the thermodynamic equation is a function of 3D divergence and of specific heat capacity of air taken at constant volume, replacing the classical conversion term depending on the pressure vertical velocity ω (thus indirectly on the vertical integral of horizontal - 2D - divergence taken for the column of air above the point) and on specific heat capacity of air taken at constant pressure.
- The mass continuity equation does not change its form; it becomes the prognostic equation for surface hydrostatic pressure. Similarly the functional form of vertical advection terms is left unchanged.

As shown in *Laprise* (1992) if one sets $p = \pi$ (i.e. the atmosphere is supposed to be in hydrostatic equilibrium, or in other words the hydrostatic hypothesis is valid), the elastic equation set is naturally reduced to the HPE. Coming back to the elastic equations, two additional prognostic variables need to be defined, based on vertical velocity $w = dz/dt$ and on true pressure p. In ARPEGE/ALADIN, their concrete definition was driven by very pragmatic reasons, coming from the linear model analysis used to derive a semi-implicit marching scheme and from the chosen way for the vertical grid staggering. The first simplification of the pressure variable is suggested by *Laprise* (1992) and it consists in predicting the non-hydrostatic pressure departure $p - \pi$ rather than pressure p to avoid cancellation of two relatively big forcing terms (gravity term and vertical pressure gradient term) in the vertical momentum equation. The final definition of the pressure variable is

$$\hat{\mathcal{P}} = \frac{p - \pi}{\pi^*} \tag{3}$$

where π^* is a basic state pressure, used to linearise the equations for finding a proper linear model to apply the semi-implicit scheme. As stated above, the reduction by π^* is comfortable for the introduction of the semi-implicit scheme. For similar reasons, the final form of the vertical velocity variable is

$$\hat{d} = -g \frac{\rho^*}{m^*} \frac{\partial w}{\partial \eta} \tag{4}$$

where the new variable $\hat{d}$ is called "pseudo vertical divergence" and m^* is a short-hand notation for $\partial \pi^* / \partial \eta$. The meaning of the "pseudo" in the variable name is that, except for the vertical derivative $\partial w / \partial \eta$, the scaling is done with the help of basic state variables, denoted by $*$. As it has already been mentioned, these definitions are fitting as well to the existing staggering of the vertical grid, see Figure (2), where all the prognostic variables and forcing terms are defined in the vertical layers (so called full-levels) while the coordinate pressure and all types of vertical velocities are defined at the layer interfaces (so called half-levels). Taking this into account, it is clear that the two new variables $\hat{\mathcal{P}}$ and $\hat{d}$ should be defined as well at the full-levels and that the vertical velocity w finds its natural definition at the interfaces.

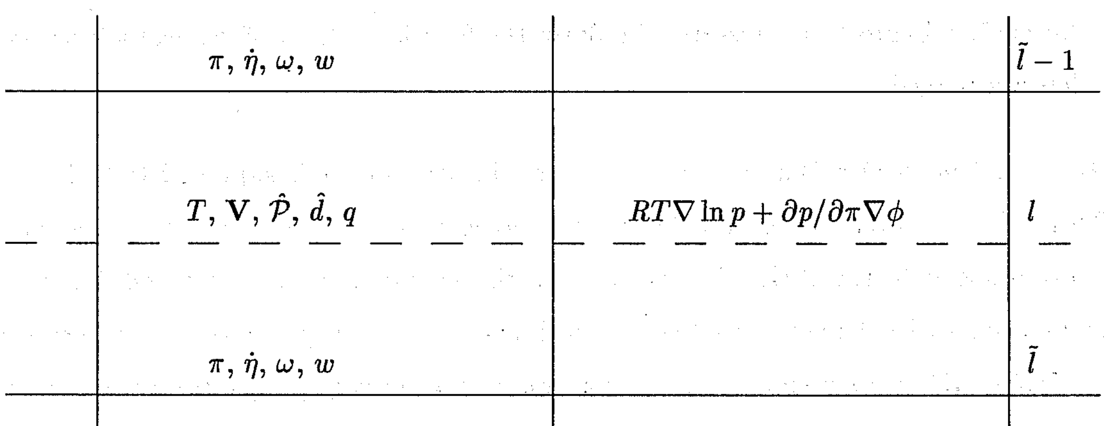

Fig. 2: Vertical grid staggering in *Simmons and Burridge* (1981) vertical discretization scheme extended to the elastic case. The numbering of levels is from the top to the bottom, the interfaces' index is denoted by a tilde.

Another interesting thing is to see the properties of the vertical discretization scheme and to examine if these properties are still kept when extending the HPE set to the elastic equations set.

As shown in *Arakawa and Suarez* (1983) -AS83-, the SB81 vertical discretization scheme is one of those few having nice conservation properties, i.e. conservation of energy and of angular momentum and, under some complementary choices, conservation of integral of potential temperature (being achieved for adiabatic atmospheres). Further, as discussed in AS83, to minimise the error in the pressure gradient force evaluation, there are two interesting choices to integrate geopotential within the family of schemes to which SB81 and AS83 belong:

- Geopotential is a function of pressure logarithm (choice done in SB81), concretely, the geopotential increments are computed between the interfaces:

$$\phi_{\tilde{l}} - \phi_{\tilde{l}-1} = -RT_l\delta_l \tag{5}$$

  where l denotes the full-level index, $\tilde{l}$ denotes the half- level index and δ_l denotes the logarithmic pressure thickness of the layer, computed as $\ln(p_{\tilde{l}}/p_{\tilde{l}-1})$ in SB81.

- Geopotential is a function of Exner pressure P (choice done in AS83), when the geopotential increments are computed across layers, between the full-levels:

$$\phi_l - \phi_{l-1} = C_p\theta_{\tilde{l}-1}(P_{l+1} - P_l) \tag{6}$$

Fortunately enough, the SB81 scheme based on hydrostatic pressure co-ordinate keeps the required conservation properties in its elastic extension, as shown in *Bubnová et al.*, 1995. There is even an additional property needed within the application of the semi-implicit scheme for the elastic dynamics, which can be achieved by changing the approximation of logarithmic pressure increments δ_l without touching the validity of the approach. The new approximation of δ_l is discussed in more details in the section on the semi-implicit scheme but since all the relationships using δ_l in the vertical discretization scheme fortunately remain untouched, the original properties of SB81 are not violated.

In both SB81 and AS83 schemes the horizontal pressure gradient (linked to the horizontal wind) and temperature (respectively potential temperature) are defined at the same levels (full-levels). This way of staggering is called the Lorenz grid and one of its properties is the existence of a spurious temperature mode which cannot be seen - and hence controlled - by the thermal wind equation: exact compensation of alternate positive and negative departures from a "regular" vertical temperature profile can indeed lead to an undetected "kernel" when integrating the barometric formula with one more element of (T,p) input than of ϕ output. This problem of the Lorenz grid is shown in detail for example in *Hollingsworth* (1995) and it may be avoided by another choice of staggering called the Charney-Phillips grid, when temperature and horizontal wind are not defined at the same levels but alternate in the vertical. However, this change of variable's staggering requires a careful re-examination of all above-mentioned desirable properties of the vertical discretization scheme. Furthermore it is not well adapted to the spectral computation framework (with its advantageous A-type horizontal grid) and to the interface between dynamics and physical forcing, for instance when the computation of Richardson numbers require to know momentum and heat fluxes at the same (half) levels. In the case of the Lorenz grid, while the spurious vertical temperature mode could in principle grow exponentially under any random forcing, practice shows that the damaging effects remain relatively bounded, thanks to all irreversible damping effects of the physical packages. Those are the reasons why this issue will not be further considered here, even if it surely remains worth of future investigations.

Another problem is common to the schemes using terrain-following co-ordinates, regarding the accuracy of pressure gradient computations. There are two terms computed separately to obtain the pressure gradient force and each of them is evaluated with some truncation error which may not cancel and result in a large error, especially over steep slopes. As mentioned above, this problem is studied in both SB81 and AS83 schemes and thanks to the careful design (typically the use of some remaining degrees of freedom) the error is minimised and in some special cases it vanishes. However, there exists the other alternative of using a so called step-mountain coordinate (*Mesinger et al.*, 1988) where the orography is specified with help of parallelepipedic grid boxes and where the hybrid vertical coordinate is defined in such a way that the coordinate surfaces are piecewise quasi-horizontal while jumping from one level to the next when appropriate. Though the pressure gradient force is computed still from two terms, the removal of coordinate surface slopes helps to reduce significantly a large part of the error. On the other hand this system cannot be used for instance together with a spectral method for obvious reasons coming from the orography definition: a spectral fit would introduce a lot of Gibbs waves near the coordinate "jumps". A special care must be devoted as well to the boundary conditions at the lateral walls of the orography, which represents additional difficulties for more general applications of the step-mountain coordinate.

3.2 Semi-Implicit Scheme

There is a well-known problem of the control of acoustic waves, which are allowed as a solution of the elastic equations. These waves propagate very fast also in the vertical where the grid size is much finer compared to the horizontal scale (for instance the order of grid size is 10 m near the ground in vertical and 10 000 m in horizontal in current NWP meso-scale models). Thus CFL criteria of a simple explicit scheme would lead to the order of a tenth of second for the time-steps. The choice to be done here to obtain an efficient integration scheme comparable to those used for the HPE is either to use an anelastic equation set (acoustic waves are filtered out) or a numerical control of the acoustic waves. If opting to keep the elastic equations, there are two possible ways of such a control: (i) by introducing either a so-called split-explicit scheme (the terms responsible for the acoustic waves propagation are integrated separately with small enough time-steps) or (ii) by implementing a semi-implicit scheme consisting in an implicit correction of the crucial acoustic wave generating terms with respect to the explicit scheme. In order to still increase the efficiency of the numerical scheme, a semi-implicit marching scheme may be combined with a semi-Lagrangian treatment of advection. This latter choice for integration of the elastic equations seems to be very attractive since there are more and more examples of non-hydrostatic fully compressible (elastic) semi-implicit semi-lagrangian (SISL) models; we may refer some of them:

- MC^2 model (Gal-Chen and Somerville type of coordinate with Charney-Phillips staggering, grid-point model, *Tanguay et al.* (1990) or *Laprise et al.* (1997));
- ARPEGE/ALADIN model (Laprise coordinate with Lorenz staggering, spectral model, *Bubnová et al.* (1995));
- Next Generation UK Model (Gal-Chen and Somerville coordinate with Charney-Phillips staggering, grid-point model, *Cullen et al.* (1997));
- Global NH SISL model (Gal-Chen and Somerville coordinate with Charney-Phillips staggering, grid-point model, *Qian, Semazzi and Scroggs* (1998)).

However, the implementation of a semi-implicit (SI) scheme is not always straightforward, especially for more complex systems of equations with orography. Taking ARPEGE/ALADIN as an example, the SI scheme has already been there to control numerically the propagation of gravity waves in the HPE set and the goal was to extend it to control both gravity and acoustic waves. The SI scheme implementation in ARPEGE/ALADIN is done very classically, in the form of corrections, which in the leap-frog case reads:

$$\frac{X^+ - X^-}{2\Delta t} = \mathcal{M}X^0 + \beta\Lambda\left(\frac{X^+ + X^-}{2} - X^0\right) \tag{7}$$

where X^+, X^0, X^- are the model states at the time-steps $t + \Delta t$, t and $t - \Delta t$ respectively, $\mathcal{M}$ denotes an explicit model step and Λ is the linear model describing the propagation of the most rapid modes (gravity and acoustic ones in the case of the elastic equations), which phase speeds need to be slowed down back below CFL speed by the semi-implicit correction. In case the linear model does not control sufficiently all the rapid modes, the correction does not stabilise the scheme for time-steps longer than allowed by the CFL condition relative to these modes. The explicit scheme may be obtained by setting the correction coefficient β to zero. The SI correction starts to have stabilising effects for $\beta > 0.5$, while its usual setting in practice is $\beta = 1$. When putting all terms containing the future model state on the left-hand side, one gets a Helmholtz type of equation to be solved for X^+:

$$(\mathbf{I} - \beta\Delta t\Lambda)\, X^+ = X^- + 2\Delta t \mathcal{M} X^0 + \beta\Delta t\Lambda \left(X^- - 2X^0\right) \tag{8}$$

In the case of a spectral model, this equation is very easily solved in spectral space. On the other hand, since the computational operations with spectral series have to be kept linear, there are limitations imposed on the choice of a basic state around which the equations are linearised (no dependency on horizontal coordinates, for example) and hence, on coefficients of the solver. The basic state used in ARPEGE/ALADIN is indeed very simple: it is a dry, isothermal, resting atmosphere with all horizontal gradients equal to zero. Thus it is fully determined only by its temperature T^* and by the constant value of surface pressure, respective by the one of hydrostatic surface pressure π_s^* in our case of the elastic dynamics. As in the previous section, the basic state is denoted by a star. The obtained linear set of the elastic equations is then simple as well:

$$\frac{\partial D}{\partial t} = -R\int_{\pi^*}^{\pi_s^*} \frac{\nabla^2 T}{\pi^*} d\pi^* + RT^* \int_{\pi^*}^{\pi_s^*} \frac{\nabla^2 \hat{\mathcal{P}}}{\pi^*} d\pi^* - RT^*\nabla^2\hat{\mathcal{P}} - RT^* \frac{\nabla^2 \pi_s}{\pi_s^*} \tag{9}$$

$$\frac{\partial \hat{d}}{\partial t} = -\frac{g^2}{RT^*}\pi^* \frac{\partial}{\partial \pi^*}\left[\frac{\partial}{\partial \pi^*}\left(\pi^*\hat{\mathcal{P}}\right)\right] \tag{10}$$

$$\frac{\partial T}{\partial t} = -\frac{RT^*}{C_v}\left(D + \hat{d}\right) \tag{11}$$

$$\frac{\partial \hat{\mathcal{P}}}{\partial t} = -\frac{C_p}{C_v}\left(D + \hat{d}\right) + \frac{1}{\pi^*}\int_0^{\pi^*} D d\pi^* \tag{12}$$

$$\frac{\partial \pi_s}{\partial t} = -\int_0^{\pi_s^*} D d\pi^* \tag{13}$$

where the vertical integrals and derivatives are written in their continous and not yet discretized forms. Applying the elimination method, the system is reduced to its structure equation which determines the shape of the solver. While it is rather easy to proceed with the elimination for the case of continous vertical operators, the same path is not necessarily followed when using the discrete form of these operators. For example, the discretization of vertical integrals appearing in the equations (9) and (12) can be taken from the HPE case, since these integrals are just the same. However a problem occurs for the double vertical integral obtained when substituting $\partial\hat{\mathcal{P}}/\partial t$ in the partial time derivative of (9) using (12). In the continuous case this double integration may be transformed by integrating it "per partes":

$$\int_{\pi}^{\pi_s} \frac{1}{\pi^2} \int_0^{\pi} (\,)d\pi' d\pi'' = \int_{\pi}^{\pi_s} \frac{1}{\pi} (\,)d\pi' + \frac{1}{\pi} \int_0^{\pi} (\,)d\pi' - \frac{1}{\pi_s} \int_0^{\pi_s} (\,)d\pi' \tag{14}$$

a nice cancellation of all vertical integrals ensuing. In case the discrete form of the structure equation and hence the solver should stay simple, some way has to be found to make the discrete integrals follow the per partes integration rule. As shown in *Bubnová et al.* (1995) this can be achieved in the SB81 case by changing the approximation of the logarithmic pressure thickness of layers δ_l from

$$(\delta_{\tilde{l}}^*) \;=\; \ln \frac{\pi_{\tilde{l}}^*}{\pi_{\tilde{l}-1}^*} \tag{15}$$

to

$$(\delta_{\tilde{l}}^*) \;=\; \frac{\pi_{\tilde{l}}^* - \pi_{\tilde{l}-1}^*}{\sqrt{\pi_{\tilde{l}-1}^* \pi_{\tilde{l}}^*}} \tag{16}$$

This new approximation has to be used in both linear and non-linear parts of the model. It is a bit less precise, since $d\ln\pi$ is in fact replaced by $d\pi/\pi$, a change which suggests the definition of the hydrostatic pressure at full-levels as:

$$(\pi_{\tilde{l}}^*) \;=\; \sqrt{\pi_{\tilde{l}-1}^* \pi_{\tilde{l}}^*} \tag{17}$$

It should be noted that in the SB81 scheme there was no need to specify pressure at the full-levels. In the present scheme as applied to the elastic dynamics it becomes necessary to specify the values of coordinate pressure π at the full-levels due to the definition of $\hat{\mathcal{P}}$ there. Hence, the knowledge of π_l becomes necessary for computation of some terms in the dynamics. The expression (17) satisfying the rule of integration per partes is used as well in the non-linear model to determine the full-levels' hydrostatic pressure.

Beside this problem of vertical integrals a discrete form of the vertical Laplacian operator appearing in the Equation (10) had to be found. Similarly to the case of the per partes rule, a combination of the vertical Laplacian and vertical integrals should lead to cancellation with a simple constant number as a result. However, in the discrete form it would require that the full-level pressures π_l^* would be equal to the half-level pressures $\pi_{\tilde{l}}^*$. Not only this condition is not very natural but it even cannot be fulfilled since it is incompatible with any possible nontrivial choice of the coefficients of vertical operators. Therefore a necessary degree of freedom is missing in the system to satisfy all the desirable constraints. Because of this, the discrete form of the structure equation does not reflect exactly its continuous counterpart. The obtained shape of the solver appearing in (8) and corresponding to the discrete form of the structure equation then reads:

$$\mathbf{I} - \frac{c_s^2}{H^2}\beta^2\Delta t^2\mathbf{L}^* - \nabla^2\mathbf{m}^2 c_s^2\beta^2\Delta t^2 - \nabla^2\mathbf{m}^2 c_s^2 N^2\beta^4\Delta t^4\left(\mathbf{I} + \mathbf{L}^*\mathbf{Q}^*\right)$$

where c_s is the speed of the sound in the basic state atmosphere, H is its equivalent height, N is the basic state Brunt-Väisälä frequency, $\mathbf{m}$ is the maximum value of map factor over the integration domain, $\mathbf{L}^*$ is the discrete vertical Laplacian operator and $\mathbf{Q}^*$ is a diagonal matrix, which appears due to the above mentioned non-cancellation of discrete operators. In the ideal case $\mathbf{Q}^*$ would be zero. In reality the diagonal elements of $\mathbf{Q}^*$ are very small, equal to:

$$q_{(1,1)} = 0 \tag{18}$$

$$q_{(l,l)} = \delta_l - 2\alpha_l \tag{19}$$

where α_l is the logarithmic pressure thickness taken between the full-level l and the half-level $\tilde{l}$ as introduced in SB81, see Figure (3). The product $\mathbf{L}^*\mathbf{Q}^*$ is the residuum obtained after performing the elimination to get the structure equation and the tridiagonal matrix $\mathbf{I}+\mathbf{L}^*\mathbf{Q}^*$ may be interpreted as a locally averaging operator in the vertical.

Since $\mathbf{L}^*$ is a tridiagonal matrix, the matrix of the whole solver fortunately remains tridiagonal as well, despite the $\mathbf{Q}^*$ term. The solver has then a very simple shape and like for the hydrostatic case, it is easily applied to the spectral coefficients. However, as it was revealed by experiments with the full non-hydrostatic non- linear model, there is a crucial difference between the hydrostatic and elastic cases: while in the hydrostatic case the gravity wave terms just contain the divergence of horizontal wind (2D) which is then fully controlled in the linear model, this is not the case for the three-dimensional (3D) divergence in the case of elastic dynamics. The 3D divergence has got its linear and non-linear part in the hydrostatic-pressure terrain-following coordinate (3D grid-boxes are not orthogonal in the transformed coordinate):

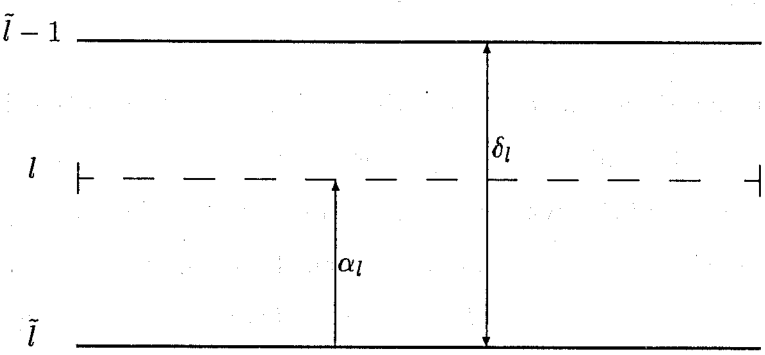

Fig. 3: Pressure logarithmic thicknesses δ_l of the layer l and α taken between the interface $\tilde{l}$ and the full-level l.

$$D_3 \;=\; D + \frac{p}{RT}\nabla\phi \bullet \left(\frac{\partial \mathbf{V}}{\partial \pi}\right) + \hat{d}\frac{\partial \phi^*}{\partial \phi} \tag{20}$$

The linear part of D_3 is just equal to $D + \hat{d}$ and the non-linear part escapes the semi-implicit correction. This is why the integration of the non-linear model is basically unstable for time-steps longer than allowed by the explicit scheme. The way to correct this insufficiency is to add a semi-implicit type of treatment to the non-linear part of D_3 denoted as Y:

$$Y = \frac{p}{RT}\nabla\phi \bullet \left(\frac{\partial \mathbf{V}}{\partial \pi}\right) + \hat{d}\left(\frac{\partial \phi^*}{\partial \phi} - 1\right) \tag{21}$$

The treatment consists in adding corrective terms proportional to the classical form

$$\beta \Delta t (Y^+ + Y^- - 2Y^0)$$

to the right-hand side of equations for temperature T and reduced non-hydrostatic pressure departure $\hat{\mathcal{P}}$ where the D_3 terms occur. Given the necessity to know Y^+, the correction is made by iterations of the solver. At the first step of the iterative procedure the solution of the Helmholtz equation is computed without this special correction, then Y^+ is computed from the first guess of the future time-step variables, the correction to the right-hand side of the Helmholtz equation is applied and the equation is solved again to provide a better guess of the prognostic variables, etc.. One iteration is usually sufficient to stabilise the integration using time-steps equivalent to those of the HPE model at the same resolution. In a spectral model the inconvenience is a non-linear computation of Y^+, which cannot be done for spectral coefficients and requires additional spectral transformation forth and back for the variables necessary to the computation of Y^+.

The formulation of the elastic dynamics in ARPEGE/ALADIN, which is the closest possible one to the HPE dynamics, enables quite clean comparisons of the simulations done with the elastic and with the hydrostatic versions of the dynamics. On Figure (4) we see vertical cross-sections showing the generalised vertical velocity $\omega = dp/dt$ obtained from the simulations of a squall line case done with a $7km$ mesh (*Banciu, Gérard and Geleyn*, (1999)). Though the ω velocity patterns already show some differences at such a scale, much clearer non-hydrostatic effects may be seen on the field of potential temperature taken from a PYREX'90 case simulation done with a $2.5km$ mesh, see Figure (5). While the hydrostatic model simulates just an intensive hydraulic-jump type wave above the ridge of the Pyrennées, the elastic model starts to catch the observed lee-wave pattern.

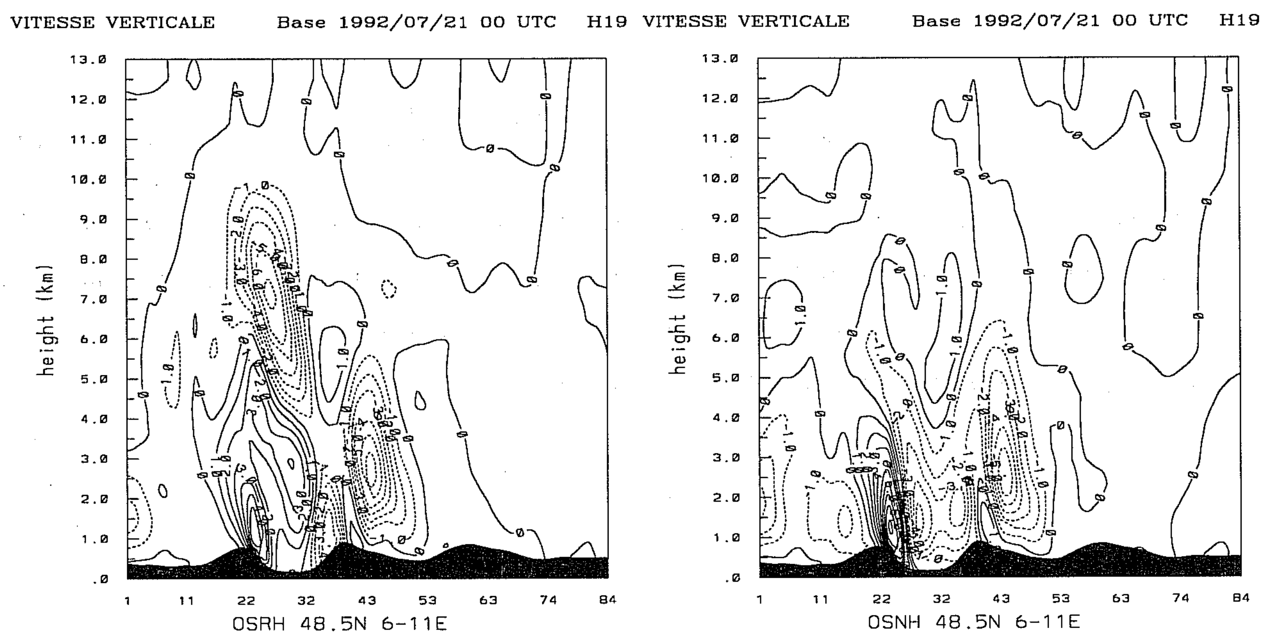

Fig. 4: Vertical velocity ω $[Pa/s]$ cross section in the simulation of a squall line case done at $7km$ mesh. The simulation with the hydrostatic model is on the left picture, while the one with the elastic model is on the right picture.

3.3 Lower Boundary Condition

It happens that there exists a necessity to determine some terms at the bottom and top boundaries of the model domain to complete the conditions coming from the generalised coordinate velocity $\dot{\eta} = 0$ at these boundaries. A method often employed for the bottom boundary problem in the non-hydrostatic case is called "free-slip" condition, which requires the potential vorticity to be zero

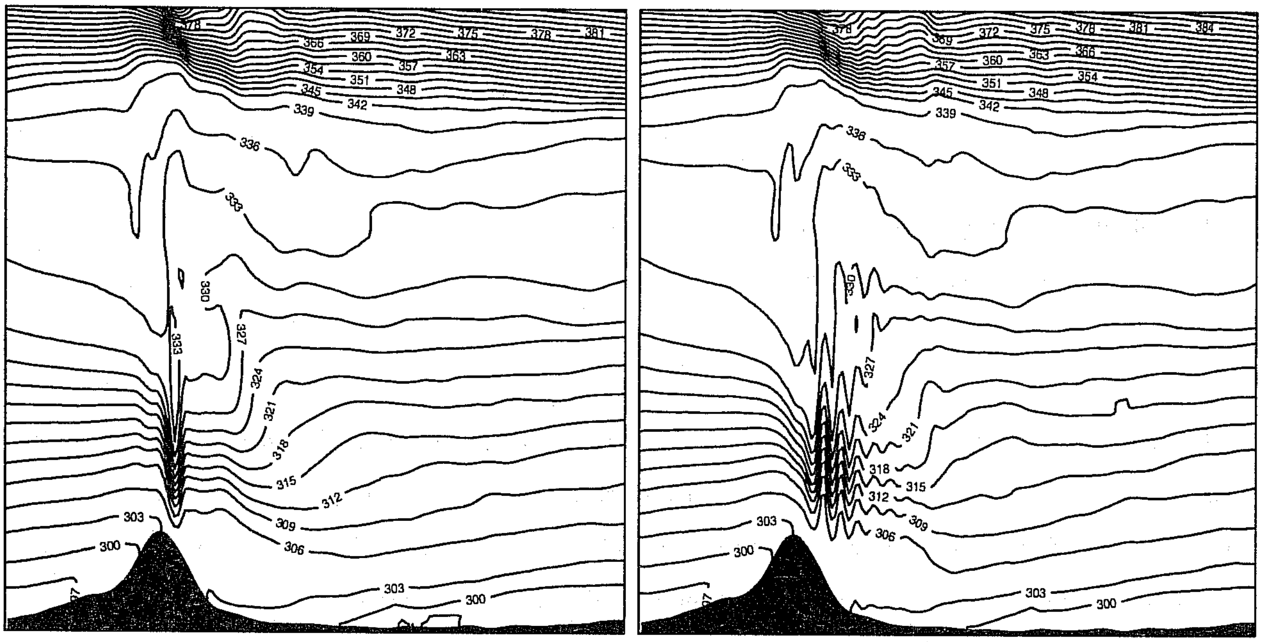

Fig. 5: 6 hour forecast of the potential temperature θ $[K]$ in the PYREX'90 case done at $2.5km$ mesh. The simulation with the hydrostatic model is on the left picture, while the one with the elastic model is on the right picture.

at the bottom surface; however it is very often understood and applied in its simplest form: zero vertical gradient of the horizontal wind at the bottom boundary. Though this simple form may be quite satisfactory, there are surely problems with it in the transformed coordinate system due to the latter's non-orthogonality, especially over steep mountains. Taking again ARPEGE/ALADIN as an example, the specification of the horizontal wind is necessary to determine several other needed terms. The above mentioned simplest form of the "free-slip" condition reads:

$$\mathbf{V}_{\tilde{L}} = \mathbf{V}_L \tag{22}$$

where the index L denotes the lowest full-level of the model where wind is defined as a prognostic variable and the index $\tilde{L}$ denotes the lowest half-level, i.e. the model bottom surface. Then from the kinematic rule the following quantities needed in the elastic model may be determined:

– vertical velocity $w_{\tilde{L}}$ at the bottom:

$$w_{\tilde{L}} = \frac{1}{g}\mathbf{V}_{\tilde{L}} \bullet \nabla\phi_{\tilde{L}} \tag{23}$$

– horizontal gradient of the vertical velocity $w_{\tilde{L}}$:

$$\nabla w_{\tilde{L}} = \frac{1}{g}\nabla\left(\mathbf{V}_{\tilde{L}} \bullet \nabla\phi_{\tilde{L}}\right) \tag{24}$$

– vertical acceleration $\dot{w}_{\tilde{L}}$ at the bottom:

$$\dot{w}_{\tilde{L}} = \frac{1}{g}\frac{d}{dt}\left(\mathbf{V}_{\tilde{L}} \bullet \nabla\phi_{\tilde{L}}\right) \tag{25}$$

The bottom boundary condition for the horizontal wind $\mathbf{V}$ is needed as well when evaluating the term containing the vertical derivative of $\mathbf{V}$ appearing in the D_3 term (e.g. Equation (20)):

$$\rho\nabla\phi \bullet \left(\frac{\partial \mathbf{V}}{\partial \pi}\right)$$

where, in opposition to the terms of vertical advection of $\mathbf{V}$, no multiplication by $\dot{\eta}$ simplifies the evaluation at the boundaries. For all these computations the simplest form of the "free-slip" boundary condition has been used up to now in ARPEGE/ALADIN. However, as mentioned above, there are problems due to the non-orthogonality of the transformed coordinate system. This is for example well shown by *Satomura* (1989), who compared two simulations of a 2D idealised mountain flow; the first one was done using the classical terrain-following coordinate while the second one was done using an orthogonal numerically generated grid. As it can be easily seen on Figure (6), there is a spurious wave pattern in the vertical velocity w above the mountain in the case of the terrain-following coordinate. A very similar (even if small) pathological pattern may be found in the idealised mountain wave simulations with the elastic version of ARPEGE/ALADIN, as shown on Figure (7).

However, the numerical generation of orthogonal grids is a very complex and expensive procedure and in fact it turns to be unpracticable for 3D systems and would anyhow create unsolvable problems in the spectral framework. In order to remove at least partially the problem of non-orthogonality, a more complex form of the free-slip bottom boundary condition has been under development in ARPEGE/ALADIN, a bit closer to the ideal one. Instead of simply taking the zero gradient of $\mathbf{V}$ along the vertical coordinate η, the projection of 3D vorticity to the axis normal to the surface should be zero, from which a correction to $\mathbf{V}_L$ is computed, as illustrated on Figure (8).

In order to compute the wind $\tilde{\mathbf{V}}_L$ in this case, the following simplification assumptions are still needed:

- divergence D_3 is zero between the surface and the lowest full-level;
- $\nabla\left(\mathbf{V}_L \bullet \nabla\phi_{\tilde{L}}\right) = \nabla\left(\mathbf{V}_{\tilde{L}} \bullet \nabla\phi_{\tilde{L}}\right)$
- local time derivative of the difference $\mathbf{V}_L - \mathbf{V}_{\tilde{L}}$ is zero.

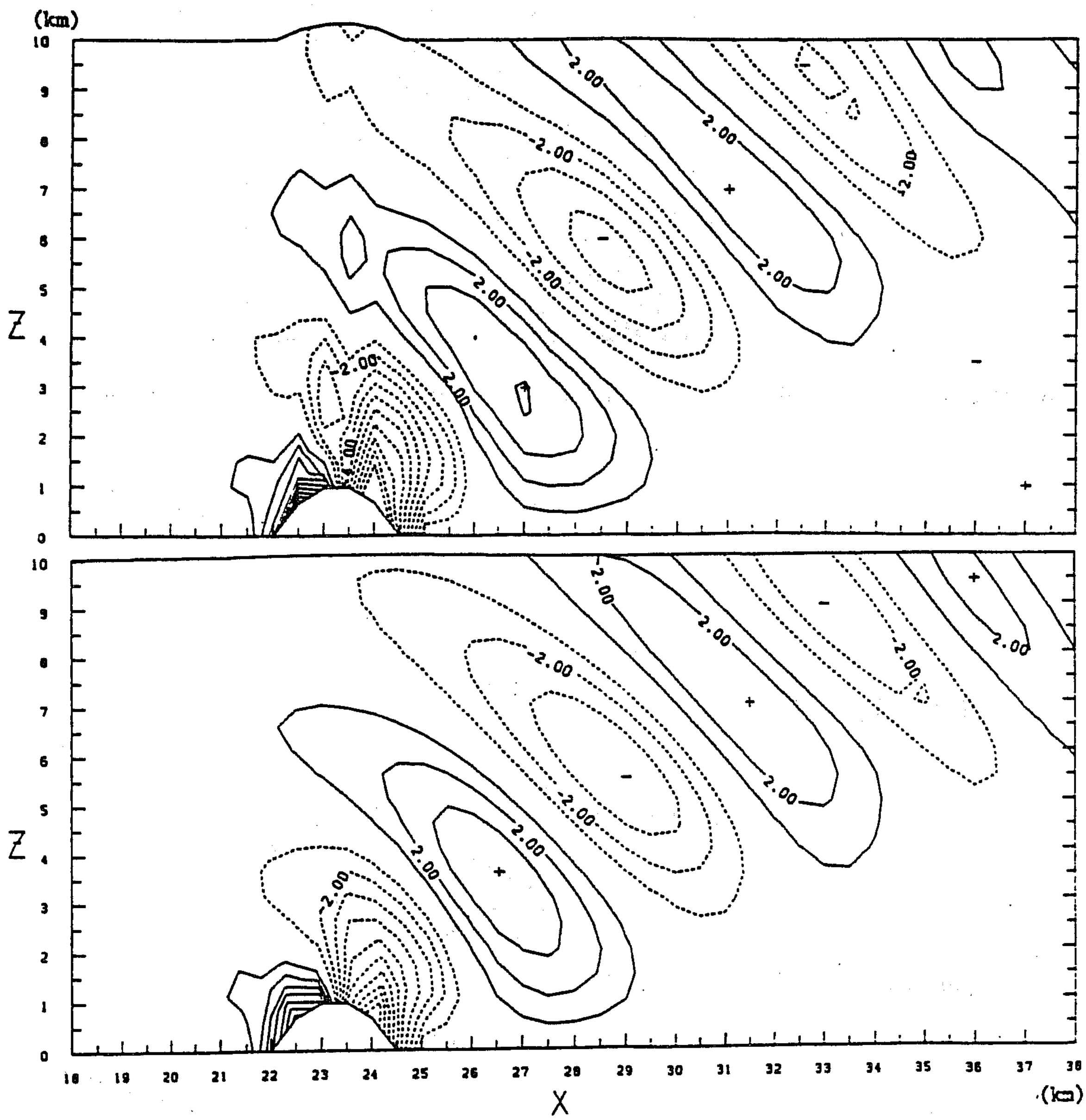

Fig. 6: After *Satomura* (1989). The simulation with a terrain-following co-ordinate is on the upper picture, the simulation with an orthogonal grid is on the bottom picture.

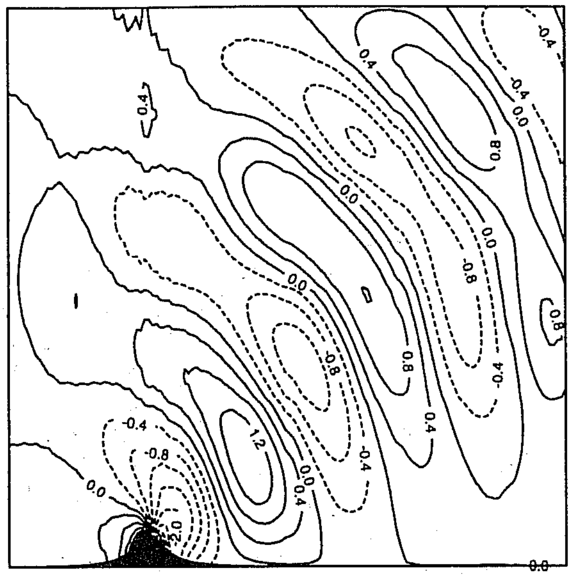

Fig. 7: Nonlinear non-hydrostatic flow simulated with ARPEGE/ALADIN: vertical velocity w $[m/s]$.

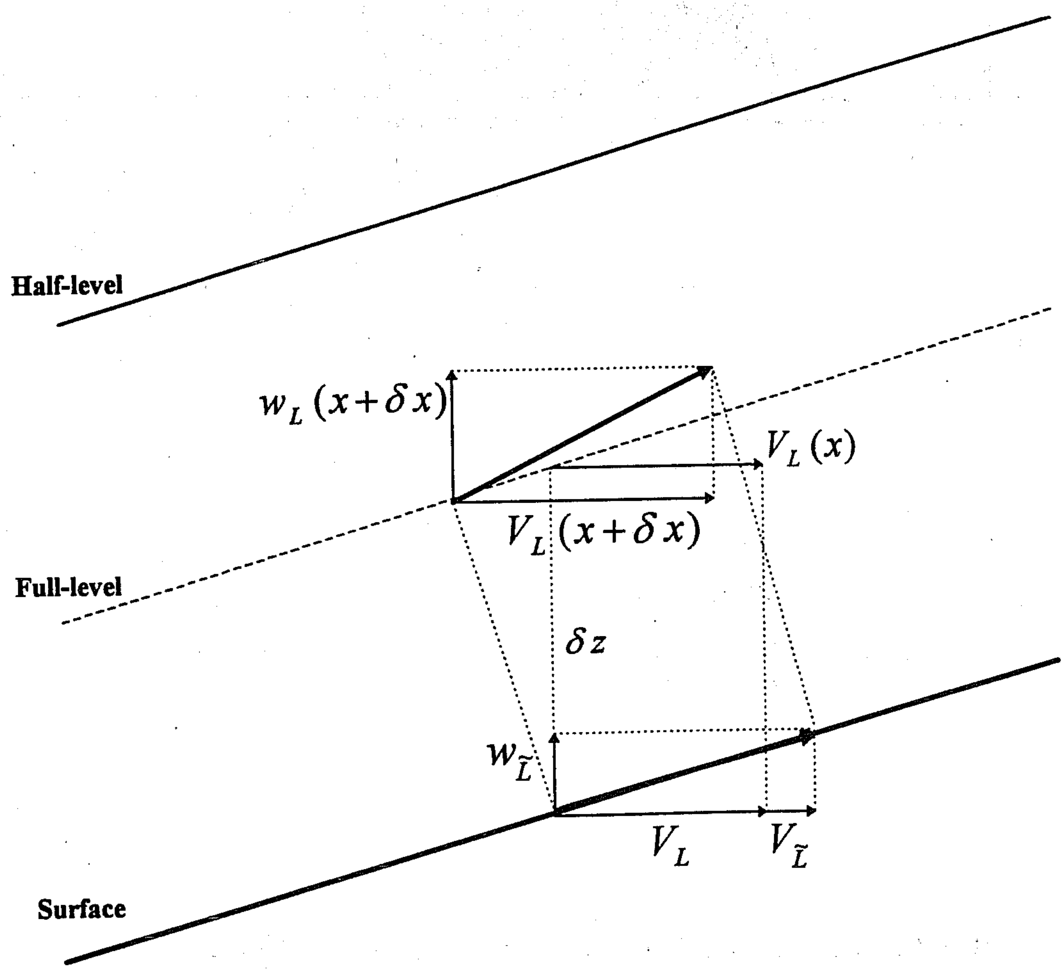

Fig. 8: Reformulation of the free-slip boundary condition: illustration to the new way of computation of the surface horizontal wind denoted as $\mathbf{V}_{\tilde{L}}$.

When denoting by α and β the horizontal derivatives of orography with respect to x and y respectively, we obtain the following relationship for the corrections of wind components $\Delta u = u_L - \tilde{u}_L$ and $\Delta v = v_L - \tilde{v}_L$:

$$\Delta u = \frac{\Delta z}{1+\alpha^2+\beta^2}\left(\frac{\partial(\alpha u)}{\partial x} + \frac{\partial(\alpha v)}{\partial y} + \alpha\frac{\partial u}{\partial x} + \beta\frac{\partial u}{\partial y}\right) \quad (26)$$

$$\Delta v = \frac{\Delta z}{1+\alpha^2+\beta^2}\left(\frac{\partial(\beta u)}{\partial x} + \frac{\partial(\beta v)}{\partial y} + \alpha\frac{\partial v}{\partial x} + \beta\frac{\partial v}{\partial y}\right) \quad (27)$$

where Δz is taken between the surface and the lowest full-level along the vertical axis. This correction of the surface horizontal wind has to be taken into account also in Equation (25) for the vertical acceleration $\dot{w}_{\tilde{L}}$ at the surface. Till now (for technical reasons) there has been no experiment done with a 2D vertical plane version of ARPEGE/ALADIN containing this new formulation of the free-slip bottom boundary condition. Unless a set of mountain wave simulation is done, it is not possible to make conclusions whether the precision of the elastic model gets improved or not (regarding the spurious wave pattern above the mountain and assessment of the momentum flux and drag). At least the stability of the scheme does not deteriorate as it has come out from some simulations done with the full 3D model.

4 RADIATIVE UPPER BOUNDARY CONDITION

In this section we shall deal with another development which becomes more and more interesting when increasing the model resolution. In reality this increase in resolution can be translated into getting a better description of the orography, which means to have steeper slopes and higher anisotropy of the terrain. As a consequence, gravity waves are explicitly generated in the model with more intensity and frequency. Since the commonly used rigid-lid type upper boundary conditions are reflective

$$\dot{z}_{top} = \dot{\sigma}_{top} = \dot{\eta}_{top} = 0$$

a considerable amount of the gravity waves energy reflected back to the computational domain may quite distort the solution. There are several possibilities how to diminish the problem due to the reflection of waves. First, in NWP models some kind of ad-hoc cure may be used: above a given altitude stronger horizontal diffusion towards the top of the domain is applied and thus it is damping a lot the short waves, but of course not only the reflected ones! This kind of a first-help may not be satisfactory at all for simulations done at high resolution, therefore the two following approaches may be followed:

- Introduction of an absorbing layer, so-called sponge. This is a damping as well like in the above mentioned "first-help" case, but it works on the principle of relaxation, similar to the one applied at the lateral boundaries of a LAM. However, the absorbing/relaxation layer has to be large enough, at least one third of the domain is needed to be devoted to the sponge. Thus it is a rather costly solution.

- Use of a radiative upper boundary condition. This is a much more attractive approach, since this upper boundary condition should allow to radiate the energy of gravity waves out from the model domain and inhibit its reflection downward. So in principle it combines two advantages: it should be cost-effective compared to the sponge technique; it should preserve the solution without a need to apply an overall damping.

To summarise, the radiative upper boundary condition (RUBC) is the most promising solution to the problem of the wave reflection. The requirements are the following: i) the RUBC should radiate most of the energy of gravity waves upward; ii) the RUBC should prohibit the downward energy transport (reflection); iii) the RUBC should be easy to implement and it should not cause instabilities in the model. An example of such RUBC for the Boussinesq set of equations cast in height coordinates is shown in *Klemp and Durran* (1983) and in *Bougeault* (1983). Though they did not start from exactly the same Boussinesq equations, they derived the same condition with the help of linear internal wave theory:

$$p_{top}^{m,n} = \frac{N^0 \rho_{top}}{\sqrt{[m^2 + n^2]}} w_{top}^{m,n} \tag{28}$$

this condition links spectral coefficients of pressure p and vertical velocity w (at the top of the model) with m and n denoting wave numbers (inverse of wave length) in the x, respectively y direction and N^0 denoting the Brunt-Väisälä frequency. A good feature of this RUBC is that it is local in time and that it is easily applicable despite the need of bi-Fourier transforms. As shown in *Klemp and Durran* (1983) the simple relation (28) remains efficient even for more complex models than the one for which it was derived. This kind of RUBC has then been adapted to the hybrid pressure-type terrain-following coordinate η by *Herzog* (1995):

$$\dot{\eta} \left(\frac{\partial p}{\partial \eta} \right)_{top}^{m,n} = - \frac{p_{top} \sqrt{[m^2 + n^2]}}{\sqrt{\kappa R T^0}} \phi_{top}^{m,n} \tag{29}$$

relating the spectral coefficients of the generalised vertical velocity $\dot{\eta}(\partial p / \partial \eta)$ and geopotential at the top of the model. While in Equation (28) one would have to tune the value of Brunt-Väisälä frequency for more general atmospheres, in (29) it is the value of temperature T^0 that has to be tuned on a rather empirical basis, provided that T^0 stays within realistic limits for a temperature. The RUBC given by the relationship (29) has been implemented in ARPEGE/ALADIN for both Eulerian and semi-Lagrangian advection schemes, taking advantage from the already existing spectral representation of variables (*Dvořák* (1998), personal communication). The effects of this RUBC implementation are demonstrated in simulations with a $10km$ horizontal mesh and $T^0 = 280K$, as shown on Figure (9): in comparison with the "first-help" case, the RUBC is at the same time a more efficient and structure preserving technique, since it avoids unnecessary damping.

The first results obtained with Herzog's formulation of the RUBC when implemented in ARPEGE/ALADIN are thus quite promising. We even did not witness any instability problem of the kind reported in *Herzog* (1995) leading to the limitation of the time-step length to Eulerian-type values. However, a problem of spurious noise close to the lateral boundaries appears due to the necessary biperiodic form of the fields subjected to the spectral transforms: since there are quite large differences between the lateral boundary values of ϕ on each side of the domain, a sharp gradient occurs in the extension zone. Then, due to the truncated spectra, a Gibbs phenomenon is seen in the spectrally fitted geopotential field inside the physically representative area and it is even amplified by the selective effect of the wave-number term in Equation (29). Therefore a more sophisticated implementation of the RUBC has now to be found to remove this currently blocking problem. Another issue will be the possible extension of Herzog's work to the case of the Laprise type elastic dynamics. We already know that even the straightforward application of Equation (29) to the sole gravity wave terms of ALADIN in its non-hydrostatic version leads to instability of the whole model. Theoretically speaking one ought not only to stabilise this aspect of the scheme but also to prevent reflection of acoustic waves at the top of the model. Whether it will be possible or not to obtain it by meaningfully linking the generalised vertical velocity at the top of the model to geopotential perturbations together with one of the two new prognostic variables (pressure departure and vertical velocity) remains an open question.

5 RELEASE OF THE THIN LAYER HYPOTHESIS

The thin layer hypothesis is used in the majority of atmospheric models. This hypothesis means that the atmosphere is considered to be of negligible vertical extent compared to the radius of the earth a: instead of using $a + z$ in the metric terms, just a is used in the primitive equations, for the seek of simplicity. Consequently, the parts of the Coriolis force created by vertical motions must

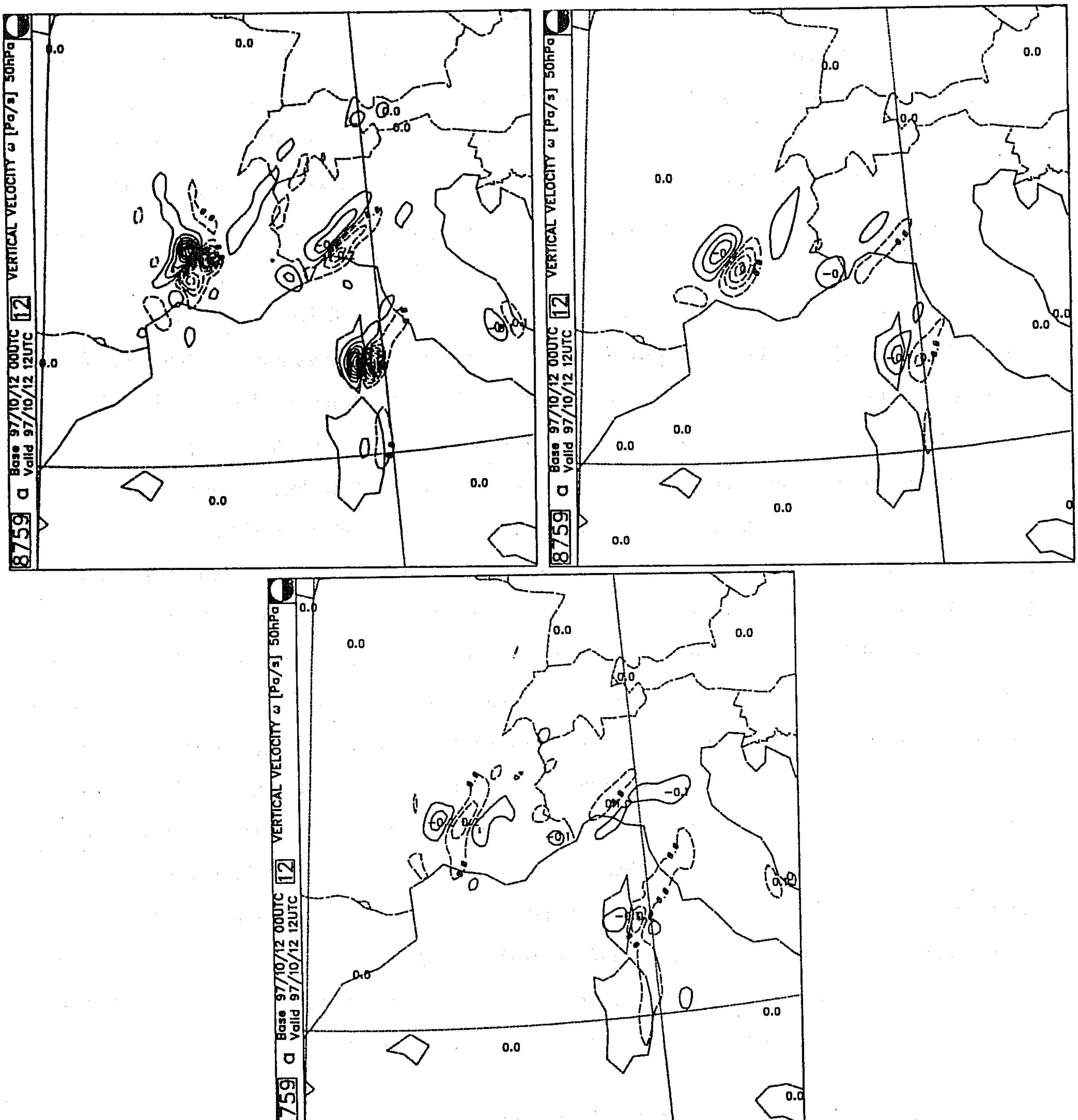

Fig. 9: Test of RUBC in ARPEGE/ALADIN on $10km$ mesh, using Eulerian advection. The reference case vertical velocity $\omega\,[Pa/s]$ at $50hPa$ level is on the upper left picture, the one using the "first-help" approach is on the upper right picture, and the one using the RUBC is on the bottom picture. The temperature T^0 was tuned to 280 K.

be omitted too, in order to conserve the angular momentum in "vertical displacements" as shown by *Phillips* (1966). But these neglected terms of the Coriolis force are just those believed to be important for fine-scale modelling (strong vertical velocities) and for climate modelling (systematic effects corresponding to the steady general circulation and its associated Hadley and Ferrel cells). Also their $\Omega \cos\phi$ form means that they anyhow become relatively important in the tropics, where the $\Omega \sin\phi$ parts tend to vanish. Furthermore, the appearance of one of these Coriolis terms in the vertical acceleration and the w dependency of the other one indicate that there is a strong link between the hydrostatic assumption and the thin layer hypothesis. For all these reasons it may become appropriate to find a way to jointly suppress both simplifications linked to the thin layer hypothesis, but in a way that preserves all otherwise important properties of the existing vertical discretization schemes.

The consistently neglected terms may be demonstrated on the following (full) momentum equation set, developed in the spherical geometry, and for othogonal x, y, z coordinates:

$$\frac{du}{dt} = 2\Omega\left(v\sin\phi - \widetilde{w\cos\phi}\right) + \frac{uv}{a+\underline{z}}\tan\phi - \frac{\mathbf{uw}}{\mathbf{a}+\underline{\mathbf{z}}} - \frac{1}{\rho\,(a+\underline{z})\cos\phi}\frac{\partial p}{\partial\lambda} \tag{30}$$

$$\frac{dv}{dt} = -2\Omega u\sin\phi + \frac{u^2}{a+\underline{z}}\tan\phi - \frac{\mathbf{vw}}{\mathbf{a}+\underline{\mathbf{z}}} - \frac{1}{\rho\,(a+\underline{z})}\frac{\partial p}{\partial\phi} \tag{31}$$

$$\widehat{\frac{dw}{dt}} = 2\Omega + \widetilde{u\cos\phi} + \frac{\mathbf{u}^2+\mathbf{v}^2}{\mathbf{a}+\underline{\mathbf{z}}} - g - \frac{1}{\rho}\frac{\partial p}{\partial z} \tag{32}$$

where the following notation is applied:

- The underlined terms are the ones neglected when applying the thin layer approximation $a >> z$;

- The terms with a tilde correspond to the height variation induced planetary angular momentum changes suppressed in connection to the thin layer approximation, considering the apparent lack of change of the distance to the Earth's centre during vertical motions;

- The bold face terms are the metric terms which have to be neglected when the thin layer hypothesis is applied;

- The acceleration term with a hat is neglected under the hydrostatic assumption.

However, it is not so obvious to reintroduce back the Coriolis terms linked to the vertical motions when keeping at the same time all desirable conservation properties of the vertical discretization scheme. A way how to reintroduce these terms should be preferably considered for both z and p type of vertical coordinates. Considering this, there exists the proposal of *White and Bromley* (1995):

- in $a + z$ terms, to replace z by:

$$z(p) = \int_p^{p^*} \frac{RT^*(p')}{gp'} dp'$$

- to replace w (except for its time derivative) by:

$$-\frac{RT^*(p)\omega}{gp}$$

- to replace T in the hydrostatic equation by:

$$T + T^*(p) \frac{2\Omega u\,(a + z(p))\,cos\phi + u^2 + v^2}{(a + z(p))\,g}$$

which means to use the hydrostatic basic state to relax the thin layer hypothesis without going back to the Navier-Stokes equations. There may be some practical difficulties within the implementation to some numerical models. For example in spectral models, the height $z(p)$ varies along η surfaces but should commute with spectral transform operators. Futhermore, it is not so obvious how to treat this problem when using the co-ordinate systems of a projection to the plane instead of the coordinate system λ, ϕ on the sphere. A special treatment of these additional Coriolis terms will also have to be worked out for the semi-Lagrangian two-time-level schemes using the advective form of the Coriolis force, etc.. All these potential problems shall be treated within the forthcoming planned implementation of the idea of *White and Bromley* (1995) in the ARPEGE/ALADIN model.

6 CONCLUSIONS

In the type of work considered here it becomes very soon obvious that nothing can be got for free (Murphy-type law): each solution brings in a new problem elsewhere and progress can only be marginal in each occasion and hence globally slow. Thus, if trying to design a minimum of strategy for progress, consistency between many "individual" choices appears as a helpful and safe guideline; the corresponding ironic motto in ALADIN related work reads: "everything is under control". Consequently, progress in this specialised part of NWP named "vertical discretization problems" has really been up to now a step by step business, even if some recent attempts at a more "compact" approach (UKMO, RPN) can now be witnessed in the framework of huge and long-term projects. Within all associated limitations to either of these strategies, the march towards the "Navier-Stokes" equations seems however to be irreversible. Whether it will lead to a reduction or an increase in the number of operationally viable solutions remains however an open question!

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REFERENCES

Arakawa, A., and M. J. Suarez, 1983: Vertical differencing of the primitive equations in sigma coordinates. Mon. Wea. Rev., 111, 34–45.

Banciu, D., L. Gérard and J.-F. Geleyn, 1999: High resolution study of a squall line. To appear in EWGLAM Newsletter, No. 28.

Bougeault, P., 1983: A non-reflective upper boundary condition for limited-height hydrostatic models. Mon. Wea. Rev., 111, 420–429.

Bubnová, R., G. Hello, P. Bénard and J.-F. Geleyn, 1995: Integration of the fully elastic equations cast in the hydrostatic pressure terrain-following coordinate in the framework of the ARPEGE/-Aladin NWP system. Mon. Wea. Rev., 123, 515–535.

Côté, J., S. Gravel, A. Méthot, A. Patoine, M. Roch and A. Staniforth, 1998: The operational CMC-MRB Global Environmental Multiscale (GEM) model. Part I: Design Considerations and Formulation. Mon. Wea. Rev., 126, 1373–1395.

Courtier, P., C. Freydier, J.-F. Geleyn, F. Rabier and M. Rochas, 1991: The Arpege project at Météo-France. ECMWF Seminar Proceedings on "Numerical methods in atmospheric models", 9-13 September 1991, vol. II, 193–231.

Cullen, M. J. P., T. Davies, M. H. Mawson, J. A. James and S. Coulter, 1997: An overview of numerical methods for the next generation UK NWP and climate model. "Numerical methods in atmospheric and oceanic modelling". The André Robert memorial volume (C. Lin, R. Laprise, H. Ritchie, Eds.); Companion volume to Atmosphere-Ocean, 425–444.

Dudhia, J., 1993: A nonhydrostatic version of the Penn State - NCAR mesoscale model: validation tests and simulations of an Atlantic cyclone and cold front. Mon. Wea. Rev., 121, 1493–1513.

Gal-Chen, T., and R. C. J. Sommerville, 1975: On the use of the coordinate transformation for the solution of the Navier-Stokes equations. J. Comput. Phys., 17, 209–228.
Herzog, H.-J., 1995: Testing a radiative upper boundary condition in a nonlinear model with hybrid vertical coordinate. Met. and Atmos. Phys., 60, 185–204.

Hollingsworth, A., 1995: A spurious mode in the "Lorenz" arrangement of ϕ and T which does not exist in the "Charney-Phillips" arrangement. ECMWF Technical Memorandum No. 211, 12pp.

Hoyer, J. M., 1987: The ECMWF spectral limited area model. ECMWF Workshop Proceedings on "Techniques for Horizontal Discretization in Numerical Prediction Models", 2-4 November 1987, 343–359.

Klemp, J. B., and D. R. Durran, 1983: An upper boundary condition permitting internal gravity wave radiation in numerical mesoscale models. Mon. Wea. Rev., 111, 430–444.

Laprise, R., 1992: The Euler equations of motion with hydrostatic pressure as an independent variable. Mon. Wea. Rev., 120, 197–207.

Laprise, R., D. Caya, G. Bergeron and M. Giguère, 1997: The formulation of the André Robert MC^2 (mesoscale compressible community) model. "Numerical methods in atmospheric and oceanic modelling". The André Robert memorial volume (C. Lin, R. Laprise, H. Ritchie, Eds.); Companion volume to Atmosphere-Ocean , 195–220.

Machenhauer, B., and J. E. Haugen, 1987: Test of a spectral limited area shallow water model with time dependent lateral boundaries conditions and combined normal mode/semi-Lagrangian time integration schemes. ECMWF Workshop Proceedings on "Techniques for Horizontal Discretization in Numerical Prediction Models", 2-4 November 1987, 361–377.

Männik, A., R. Rõõm, N. Gustafsson and K.-I. Ivarsson, 1998: Nonhydrostatic experimental version of the HIRLAM. EGS XXIII General Assembly, Nice, 20-24 April 1998, Abstracts.

Mesinger, F., Z. I. Janjić, S. Ničković, D. Gavrilov and D. G. Deaven, 1988: The step-mountain coordinate: model description and performance for cases of Alpine lee cyclogenesis and for a case of an Appalachian redevelopment. Mon. Wea. Rev., 116, 1493–1518.

Miller, M. J., and A. A. White, 1984: On the non-hydrostatic equations in pressure and sigma coordinates. Quart. J. R. Met. Soc., 110, 515–533.

Phillips, N. A., 1966: The equations of motion for a shallow rotating atmosphere and the traditional approximation. J. Atmos. Sci., 23, 626–629.

Qian, J.-H., F. H. M. Semazzi and J. S. Scroggs, 1998: A global nonhydrostatic semi-Lagrangian atmospheric model with orography. Mon. Wea. Rev., 126, 747–771.

Satomura, T., 1989: Compressible flow simulations on numerically generated grids. J. Meteor. Soc. Japan, 67, No. 3, 473–482.

Simmons, A., and D. Burridge, 1981: An energy and angular momentum conserving vertical finite-difference scheme and hybride vertical coordinates. Mon. Wea. Rev., 109, 2003–2012.

Tanguay, M., A. Robert and R. Laprise, 1990: A semi-implicit semi- lagrangian fully compressible regional forecast model. Mon. Wea. Rev., 118, 1970–1980.

Tatsumi, Y., 1986: A spectral limited-area model with time-dependent lateral boundary conditions and its application to a multi-level primitive equation model. J. Meteor. Soc. Japan, 64, 637–663.

White, A. A. and R. A. Bromley, 1995: Dynamically consistent, quasi-hydrostatic equations for global models with a complete representation of the Coriolis force. Quart. J. R. Met. Soc., 121, 399–418.