

Recent research for dynamical cores of nonhydrostatic, deep-atmosphere, unified models*

Andrew Staniforth and Nigel Wood

*Met Office, FitzRoy Road, Exeter
EX1 3PB, United Kingdom
Andrew.Staniforth@metoffice.gov.uk, Nigel.Wood@metoffice.gov.uk*

Abstract

The dynamical core is a critical element of any atmospheric model. Its governing equations must include all relevant dynamical terms, and the numerical formulae used to approximate them must be accurate, stable and efficient. Recent research at the Met Office on dynamical cores is reviewed. Aspects covered include: properties of various equation sets; vertical coordinates; semi-Lagrangian advection and conservation; trajectory computation and dynamical equivalence; horizontal and vertical discretisation; and coupling of physical parametrisations to a dynamical core.

1 Introduction

The dynamical core is a critical element of any Numerical Weather Prediction (NWP) or climate-simulation model. Essential to its performance is the form of the continuous governing equations and the numerical formulae used to approximate them. A unified approach to NWP and climate modelling over a broad range of scales imposes additional constraints.

The continuous equations need to be written in a form that allows the numerical approximation to be accurate but also highly efficient in order to meet computational schedules. In addition, the numerical formulae must be numerically stable for long time integrations. Improving the efficiency of the dynamical core whilst maintaining, or even improving, accuracy allows optimisation of the model's resolution for given computing time. This aspect will be critical to the use of higher and higher resolution models. In addition, the underlying continuous equations possess certain key conservation properties. If the numerical analogue of these equations is to be accurate it should, as far as possible, preserve such properties. Research at the Met Office has therefore been undertaken into the most appropriate form of the underlying continuous equations, and into the development and exploitation of numerical schemes with improved accuracy and robustness. The approach being taken is to continue to improve, where feasible, the existing dynamical core (Davies et al. (2004); Staniforth et al. (2004)) whilst providing a development path for its future replacement.

Given the page limitation for the papers of these proceedings, references to the literature are mostly limited to published papers of recent work carried out at the Met Office. Relevant linkages to the broader literature are however available in the papers cited herein.

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2 Governing dynamical equations and vertical coordinates

2.1 Equation sets and their properties

When designing a dynamical core, it is important to identify an appropriate set of governing dynamical equations for subsequent discretisation. The very limited computing capability available in the early days of NWP and climate modelling, together with inefficient explicit time discretisation schemes, led to the adoption of various approximations to the fully compressible fluid-dynamical equations. Today, all models still make some kind of approximation to these equations, albeit far fewer than in the past. For example, on the one hand most current *global* models are based on the hydrostatic primitive equations. As such: vertically-propagating acoustic oscillations are absent via the hydrostatic assumption (this avoids the very restrictive timestep of an explicit time discretisation of the terms responsible for their existence, but invalidates the equations for many mesoscale flows); and the shallow-atmosphere assumption is made, which is based on simple scaling arguments and implies omitting various Coriolis and metric terms for dynamical consistency. On the other hand, current *mesoscale* models also make the shallow-atmosphere assumption but do not make the hydrostatic assumption: instead many of them filter out acoustic oscillations via the anelastic approximation. So what equation set should one use for a dynamical core that is applicable at all scales?

It is highly desirable, if not essential, that the equations be dynamically consistent, i.e. that they possess conservation principles for energy, angular momentum and potential vorticity, and have a Lagrangian form of the momentum equation. Staniforth (2001) and White et al. (2004) identify four such models, which correspond to whether approximations of hydrostatic and shallow atmosphere type are, or are not, individually made (see Table 1). The case for using the deep-atmosphere equations, i.e. not making the shallow-atmosphere approximation, primarily amounts to the desirability of retaining a complete representation of the Coriolis force, including the $2\Omega \cos \phi$ terms, where Ω is Earth's rotation rate and ϕ is latitude. In White et al. (2004), it is also shown that: the spherical-geopotential approximation (which is almost always made) prohibits latitudinal variation of apparent gravity; making the shallow-atmosphere approximation implies that apparent gravity should not vary as a function of r , distance from the Earth's centre; and the deep-atmosphere equations require apparent gravity to vary as $1/r^2$.

	Deep	Shallow
Nonhydrostatic	Complete Euler equations	Nonhydrostatic primitive
Hydrostatic	Quasi-hydrostatic	Hydrostatic primitive

Table 1: Equation sets for dynamical cores using Staniforth (2001)'s nomenclature .

The strategy adopted in Davies et al. (2004) has been to avoid unnecessary approximations. Accordingly, the fully compressible, nonhydrostatic equations have been adopted, without making the shallow atmosphere approximation (note though, that the spherical geopotential approximation remains). The Met Office uses the Davies et al. (2004) dynamical core as the cornerstone of its unified model, which is used for NWP and climate applications over a broad range of spatial scales, from global to mesoscale. It is believed to be the only deep-atmosphere nonhydrostatic dynamical core currently used for terrestrial atmospheric modelling.

Normal-mode analysis provides further insight into the properties and validity of governing equation sets, e.g. by identifying which terms should be treated time implicitly for computational stability. Until recently, such studies made the shallow-atmosphere approximation and neglected the vertical variation of gravity. However the Thuburn et al. (2002a,b) studies relax these two constraints. For terrestrial parameters, Thuburn et al. (2002a) found that neither constraint has significant impact on the spatial form of the energetically significant components of most normal modes, with only slight changes (less than 1%) in frequency. However, relaxing

the shallow-atmosphere approximation does lead to significant changes in the tropical structure of internal acoustic modes, primarily due to the presence of the $2\Omega \cos \phi$ Coriolis terms - this could be important in the presence of forcing, e.g. due to tropical convection. Relaxing the shallow-atmosphere approximation also leads to nonzero vertical velocity and potential temperature fields for external acoustic and Rossby modes; in contrast, these fields are identically zero when the shallow-atmosphere approximation is made. Inclusion of realistic vertical variation in the gravitational acceleration leads to a small but systematic decrease in the magnitude of normal mode frequencies, with the largest differences found being less than 1.5%.

Further insight into the role of the $2\Omega \cos \phi$ terms, associated with the northward component of the Earth's rotation vector and a deep atmosphere, is provided in [Thuburn et al. \(2002b\)](#) by deriving normal modes for an $f - F$ plane, where $f \equiv 2\Omega \sin \phi$ and $F \equiv 2\Omega \cos \phi$ are both set constant. A particularly surprising result is that the inclusion of the F terms gives rise, in planar geometry, to an additional kind of normal mode as well as the usual Rossby, gravity, and acoustic modes. The additional modes are inertial in character, have frequency very close to f , and have extremely strong vertical tilt. These modes, and the importance of the $2\Omega \cos \phi$ terms for dynamical cores, are further examined in [Kasahara \(2003\)](#).

For a finite-difference model to represent well the behaviour of the free atmosphere, it must capture accurately the structures of the normal modes. Therefore, the structures of normal modes can have implications for the choice of prognostic variables and grid staggering. [Thuburn et al. \(2002a\)](#) concluded that in the absence of other considerations, density and temperature should be analytically eliminated in favour of pressure and potential temperature as the prognostic thermodynamic variables, since the structure of density and temperature for high vertical wavenumbers would not be accurately captured on either the horizontal velocity levels or the vertical velocity levels. Also, potential temperature and vertical velocity should be staggered in the vertical with respect to the other dynamic prognostic variables, the so-called Charney-Phillips grid.

The principal tool used to develop approximate equation sets, and to assess their validity as a function of flow regime, has been scale analysis which has proven quite subtle to apply. In [Davies et al. \(2003\)](#), it is shown that normal-mode analysis provides a useful complementary tool for assessing the validity of various anelastic, hydrostatic and pseudo-incompressible equation sets for both small- and large-scale flows, and leads to the following conclusions. Whilst of key importance for small-scale theoretical studies and process modelling, the anelastic equations are not recommended for either operational numerical weather prediction or climate simulation at any scale. The pseudo-incompressible set appears to be viable for numerical weather prediction, but only at short horizontal scales. For global nonhydrostatic modelling, only the fully compressible equations are suitable. Advances in numerical techniques in the past decade (e.g. [Tanguay et al. \(1990\)](#), [Davies et al. \(2004\)](#)), allow these to be integrated in a computationally efficient manner. Note however that stability with implicit and semi-implicit timestepping and long timesteps is achieved by spuriously retarding the fast-propagating modes responsible for the timestep limitations of explicit schemes. Should fast-propagating oscillations carry non-negligible energy for a given application, then the timestep would have to be shortened in order to properly represent the physics, and the timestep advantage over an explicit treatment might be lost.

2.2 Vertical coordinates

Once an equation set for a dynamical core has been chosen, the next issue to be addressed is a suitable choice of vertical coordinate. The hydrostatic primitive equations were reviewed and analysed in [Kasahara \(1974\)](#) using a generalised vertical coordinate, defined to be any variable which is a single-valued monotonic function of geometric height. This influential review has proven to be a valuable reference, much cited by atmospheric modellers, but is not directly applicable to nonhydrostatic equation sets nor to deep-atmosphere ones. Consequently [Staniforth & Wood \(2003\)](#) extended [Kasahara \(1974\)](#)'s analysis by: relaxing the hydrostatic and shallow-atmosphere assumptions; no longer constraining the upper boundary to be a coordinate surface, to permit more general upper boundary conditions; and examining axial angular momentum conser-

vation to determine its sensitivity to the choice of upper boundary condition. This leads to a formulation of the deep-atmosphere nonhydrostatic Euler equations using a generalised vertical coordinate which includes, as a special case, the formulation of the Met Office’s new dynamical core (Davies et al. (2004)) in a height-based terrain-following coordinate. The derived energy and axial angular momentum budgets would facilitate the development of conserving finite-difference schemes for deep-atmosphere models.

It is found for a generalised vertical coordinate that the implied principles of energy and axial angular momentum conservation (in the absence of zonal mechanical forcing and mountain torque) depend on the form of the upper boundary. In particular, for a modelled atmosphere of finite extent, global energy conservation is only obtained for a rigid lid, fixed in space and time. To additionally conserve global axial angular momentum, the height of the lid cannot vary with longitude. This result has been shown to be independent of whether the atmosphere is shallow or deep, and hydrostatic or nonhydrostatic. In particular, models that impose a material surface with constant (non-zero) pressure at the upper boundary, do not conserve total energy and axial angular momentum, although they may possess an energy-like invariant. This is consistent with, and generalises, Kasahara (1974)’s analysis for a shallow hydrostatic atmosphere. There it was demonstrated that total energy is conserved for a rigid lid in height coordinates. However it is not generally conserved for an isobaric lid in pressure coordinates, but it was shown that a pseudo-energy invariant exists instead.

Today’s atmospheric models are usually formulated in terms of terrain-following coordinates. For shallow-atmosphere hydrostatic models it is natural and convenient to use pressure as the vertical coordinate (i.e. an isobaric coordinate), which has the advantage of making the continuity equation a diagnostic relation. Laprise (1992) introduced a family of “terrain-following hydrostatic-pressure” coordinates, also referred to therein as mass coordinates. This approach is valid for nonhydrostatic shallow atmospheres, retains the acoustic modes, and leads to a diagnostic continuity equation with no need for any approximations other than those of a shallow atmosphere. A simple example is

$$\eta \equiv \frac{\pi - \pi_T}{\pi_S(\mathbf{x}) - \pi_T}, \quad (1)$$

where

$$\pi(\mathbf{x}, z, t) \equiv \pi_T + \int_z^{z_T} g\rho(\mathbf{x}, z', t) dz' \left(\Rightarrow \frac{\partial \pi}{\partial z} = -\rho g \right), \quad (2)$$

is “hydrostatic pressure”, z is geometric height, \mathbf{x} is horizontal position vector, and π_T is constant. For a hydrostatic shallow atmosphere, the η coordinate (1) reduces to the traditional coordinate $\sigma = (p - p_S) / (p_S - p_T)$, since π reduces to total pressure p in the hydrostatic limit.

Using the Staniforth & Wood (2003) analysis, Wood & Staniforth (2003) showed that Laprise (1992)’s terrain-following coordinate η , based on hydrostatic pressure π for the shallow-atmosphere Euler equations, can be generalised to the deep-atmosphere Euler equations. Whereas for a shallow atmosphere, Laprise (1992)’s terrain-following coordinate can be interpreted as being either based on “hydrostatic pressure” or on mass, the generalisation to deep atmospheres is such that the analogous quantity is based on mass and not on pressure. This is because for a deep atmosphere, the cross-sectional areal element increases with height but is constant for a shallow atmosphere, resulting in different volume elements. A consequent benefit of Wood & Staniforth (2003)’s mass-based generalisation of Laprise (1992)’s coordinate is that an existing (shallow-atmosphere) hydrostatic primitive-equations model that uses a pressure-based terrain-following vertical coordinate, could be modified for nonhydrostatic deep-atmosphere applications, without the need to substantially change the scientific and computing infrastructure in which it is embedded.

2.3 Energetics

As mentioned above, Staniforth & Wood (2003) have shown, in the absence of external momentum and thermal forcing and net surface torque, that deep (and shallow) atmospheres of finite extent are only guaranteed to

globally conserve energy and axial angular momentum if a rigid lid upper boundary condition is applied. Otherwise energy is not conserved. Elastic isobaric upper lids (i.e. the lid is a specified isobaric surface) are however popular and have merit. The presence of global invariants provides a constraint on the system which can be useful when designing effective numerical schemes. An energy-like invariant is known to exist for *shallow hydrostatic* atmospheres with an elastic isobaric lid. In Staniforth et al. (2003b), a generalisation of this invariant is derived. It is $(E + p_T/\rho)$, where $p_T = p_T(\lambda, \phi)$ is the pressure at the lid, and ρ is density. This energy-like invariant: (i) is valid independently of whether the atmosphere is assumed *deep* or *shallow*, and *hydrostatic* or *nonhydrostatic*; and (ii) subsumes previous shallow-atmosphere energy-like invariants in the atmospheric literature. Note that whilst $\rho E + p_T$ is globally conserved (in the sense that its volume integral is an invariant of the system) with an elastic isobaric lid, the true total energy ρE of the system is *not* conserved. The difference between the two, viz. the contribution of $p_T(\lambda, \phi)$, represents the work done by the stationary pressure applied at the upper surface as the volume of the atmosphere below it changes.

An important practical issue, raised but left unanswered in Staniforth et al. (2003b), is whether it is better to impose a rigid or an elastic lid for an atmosphere of finite extent. When the atmosphere undergoes heating, is it better to consider that this is done at constant pressure or at constant volume? The answer is not obvious.

3 Discretisation of the dynamical equations

3.1 Semi-Lagrangian discretisation

3.1.1 Semi-Lagrangian advection and conservation

Semi-Lagrangian (SL) schemes are widely used for the advection component of many modern operational atmospheric models due to their increased efficiency and stability compared to Eulerian schemes - see Staniforth & Côté (1991). However, a common disadvantage of interpolating SL schemes is the lack of mass and tracer conservation. Though mass conservation may not be critical for short period NWP simulations, it is very important for long period simulations such as those of climate studies. Over a long simulation period, the total mass can drift significantly if no correction is applied. Hence, SL schemes which are inherently mass conserving are desirable. The challenge is to not only achieve inherent conservation, but to do so whilst minimising the additional cost over that of a traditional interpolating SL scheme. This motivated the development of Zerroukat et al. (2002)'s Semi-Lagrangian Inherently-Conserving and Efficient (SLICE) algorithm.

There are two ingredients. The first is to rewrite the Eulerian flux form

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \quad (3)$$

where ρ is a scalar field transported by velocity \mathbf{u} , in a finite-volume Lagrangian form

$$\frac{D}{Dt} \int_{\partial V} \rho dV = 0 \Rightarrow M_a^{n+1} = M_d^n, \quad (4)$$

where ∂V is a fluid parcel or Lagrangian control volume, M_a^{n+1} is its mass at time $(n+1)\Delta t$ centred on the arrival location \mathbf{x}_a , and M_d^n its mass at time $n\Delta t$ centred on the departure location \mathbf{x}_d . The second is to adapt Purser & Leslie (1991)'s cascade remapping strategy to very efficiently decompose a two-dimensional remapping problem (from Eulerian control volumes to Lagrangian ones, or vice-versa) into a number of much-simpler one-dimensional remapping problems - see Zerroukat et al. (2002) for details. An important property of cascade remapping is that it preserves characteristics of the flow, thus minimising splitting errors. Overall, it is found that in addition to exactly conserving mass, the SLICE algorithm is also competitive with standard non-conserving semi-Lagrangian schemes from the viewpoints of both computational efficiency and accuracy.

Zerroukat et al. (2002)'s algorithm in planar geometry has been extended to spherical geometry in Zerroukat et al. (2004). It has no restriction on Courant numbers and again achieves comparable, or better accuracy, as standard non-conserving and other published conserving SL schemes over the sphere. It remains to extend the scheme to three dimensions. The algorithmic complexity has been a major design constraint with the view to achieving this extension in a straightforward and flexible way without a major computational overhead.

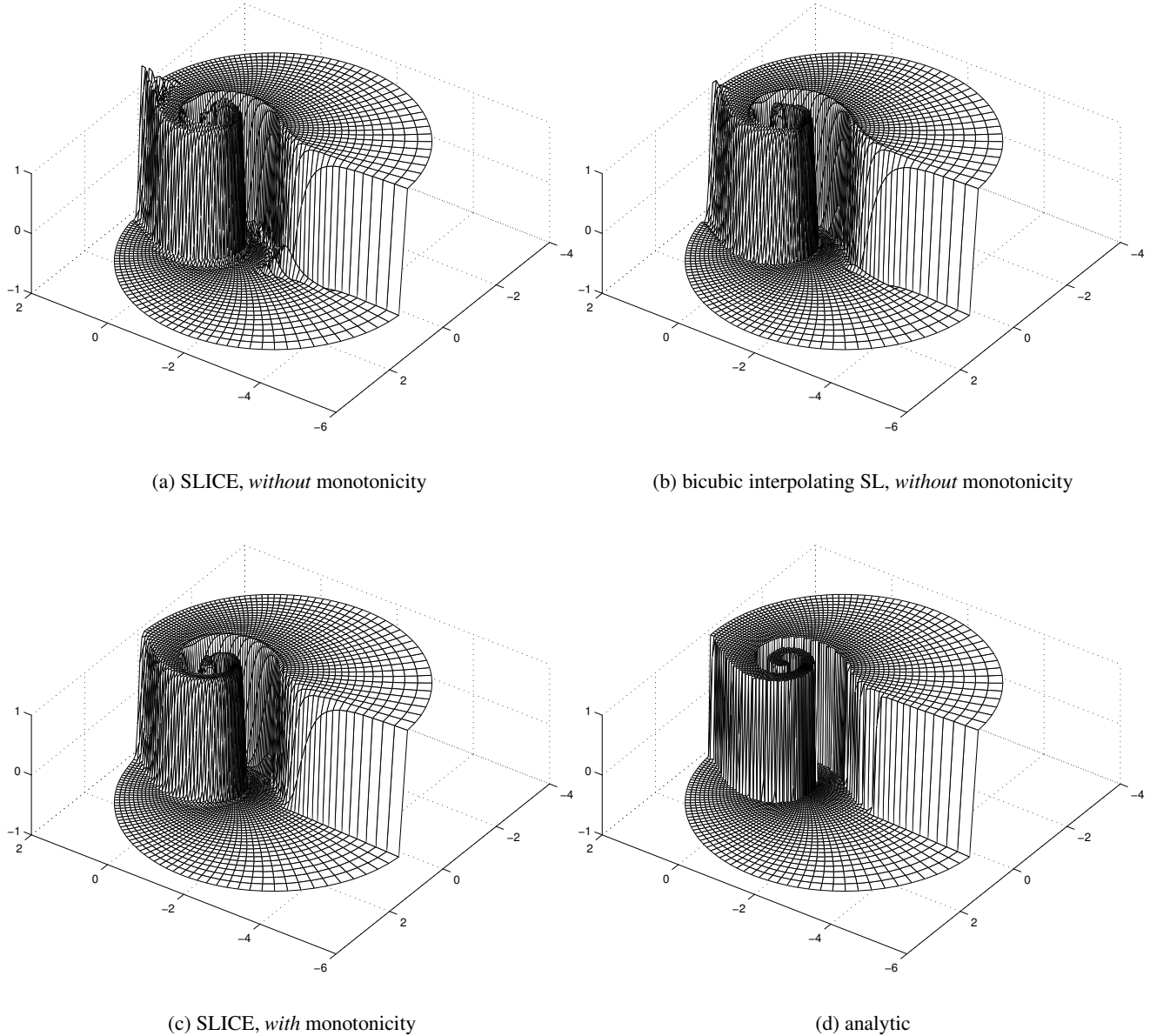


Figure 1: Solutions, projected on a tangent plane, after 64 timesteps for non smooth deformational flow on a sphere - see Zerroukat et al. (2004) and Zerroukat et al. (2005) for definition of problem and parameters.

A simple further extension of the SLICE algorithm is described in Zerroukat et al. (2005) which allows monotonicity (and positive-definiteness) to be efficiently imposed in both planar and spherical geometry. This extension operates by first identifying where monotonicity is violated (the detection stage), and by then locally reducing the order of the piecewise polynomial used in the remapping algorithm until monotonicity is re-

gained (the reduction stage). A global minimum and/or a global maximum can similarly be imposed and positive-definiteness is achieved by setting the global minimum to be zero. The resulting monotonicity scheme has been applied to various test cases. Illustrative comparative results for the challenging, non-smooth, deformational problem on the sphere are displayed in Fig. 1.

3.1.2 Trajectories and dynamical equivalence

A crucial component of any SL scheme is the computation of the trajectories (or displacement vector) via numerical approximation of $D\mathbf{x}/Dt = \mathbf{u}(\mathbf{x}, t)$, where \mathbf{x} is now three-dimensional position vector. This has an important influence on the stability and accuracy of the discretisation of the governing equations. Analytically [White \(2003\)](#) shows that the departure-point and momentum equations, given respectively by

$$\frac{D\mathbf{x}}{Dt} = \mathbf{u}(\mathbf{x}, t), \quad \frac{D\mathbf{u}}{Dt} = \mathbf{F}, \quad (5)$$

where \mathbf{F} is the local force per unit mass, convey the same information as the departure-point, angular momentum and scalar-product equations

$$\frac{D\mathbf{x}}{Dt} = \mathbf{u}(\mathbf{x}, t), \quad \frac{D(\mathbf{x} \times \mathbf{u})}{Dt} = \mathbf{x} \times \mathbf{F}, \quad \frac{D(\mathbf{x} \cdot \mathbf{u})}{Dt} = \mathbf{u}^2 + \mathbf{x} \cdot \mathbf{F}. \quad (6)$$

He refers to this property as dynamical equivalence.

The conditions under which this also holds for discrete forms of these equation sets are analysed in [White \(2003\)](#) and [Staniforth et al. \(2003a\)](#). For time-centred discretisations, [White \(2003\)](#) shows that once a rule has been chosen for the approximation of trajectory time-averages of vector and scalar products, dynamical equivalence implies a particular discrete form of the departure-point equation. For time-decentred discretisations, [Staniforth et al. \(2003a\)](#) identify two types of discretisation of the departure-point equation which preserve dynamical equivalence: some existing discretisations are then found to be approximations to them. It is also shown how to incorporate physical forcings and/or predictor-corrector dynamics into the formulation. Furthermore, two simple model problems (for solid-body rotation and wave motion) are used to provide further insight into the accuracy and stability properties of various departure-point schemes, including: dynamically-equivalent schemes; approximations to these; and several existing schemes.

3.2 Horizontal discretisation

3.2.1 Variable horizontal resolution

Like many meteorological organisations, regional forecasting and climate applications at the Met Office currently use a non-interacting or one-way nesting strategy, whereby a lower-resolution configuration of its model provides the lateral boundary conditions for a higher-resolution configuration run over a region of interest. It is well known - e.g. [Staniforth \(1997\)](#) - that there are a number of theoretical and practical problems associated with this approach. [Côté et al. \(1998\)](#) argue that it is preferable to use a fully-interacting variable-resolution (stretched grid) strategy, whereby a single global model is integrated with resolution focussed on a uniform-resolution sub-domain of a rotated lat-lon mesh. Following the successful implementation of this strategy for operational forecasting at the Canadian Meteorological Centre, the Met Office's dynamical core has therefore been generalised to variable resolution - see [Staniforth et al. \(2004\)](#) for details. Implementation of this generalisation is currently underway at the Joint Centre for Mesoscale Meteorology at Reading University.

3.2.2 Conservation and Rossby-mode propagation on the sphere

Spatial discretisations of the linearised shallow-water equations on a spherical C-grid are considered in [Thuburn & Staniforth \(2004\)](#). Constraints are derived therein that ensure conservation of mass, angular momentum and energy, and generalise previous published results to the case of non-uniform and rotated grids (but restricted to the linearised equations). Grids with either meridional velocity v stored at the poles, or azimuthal velocity u and free-surface height h , are considered. Energy conservation is shown to be problematic for grids with u and h stored at the poles. It is also found that an inappropriate averaging of the Coriolis terms leads to a misrepresentation of the Rossby modes with shortest meridional scale. The appropriate averaging is shown to not only address this problem but to be compatible with the constraints required for conservation.

3.3 Vertical discretisation

3.3.1 Analysis of a new finite-element vertical discretisation

The choice of vertical discretisation method is an important aspect of designing a dynamical core, the most popular being the use of *low-order finite differences* with a variety of vertical staggarings of dependent variables. Recently, [Untch & Hortal \(2004\)](#) have proposed a new *high-order finite-element* (FE) vertical discretisation scheme for a hydrostatic primitive equation model using a terrain-following pressure-based vertical coordinate and an unstaggered grid. This motivated its mathematical analysis in [Staniforth & Wood \(2004\)](#).

The essence of the scheme is an accurate FE algorithm for evaluating vertical integrals using either linear or cubic splines, that [Untch & Hortal \(2004\)](#) respectively denote “linear FE” and “cubic FE”. They also describe and evaluate two further methods: a finite-difference based scheme; and a “cubic collocation” scheme. The cubic collocation scheme first constructs a cubic-spline interpolant of the integrand and then analytically integrates it. [Untch & Hortal \(2004\)](#) observed that: (a) the empirically estimated truncation errors of the linear FE and cubic collocation schemes are both of order 4; (b) the measured errors of these two schemes are not only of the same order but identical (to the two significant figures given); and (c) the empirically estimated truncation error of the cubic FE scheme is of order 8. In addition to explaining these results, [Staniforth & Wood \(2004\)](#)’s analysis shows that: (d) the truncation errors of the linear FE and cubic FE schemes are respectively four and eight times smaller than those obtained by applying FE’s to the differential form of the equations instead of to the integral form; and (e) the cubic FE scheme is formally equivalent for uniform resolution to a new “heptic collocation” scheme, where a seventh-order spline is analytically integrated.

3.3.2 Discrete normal-mode analysis

Normal modes are fundamental solutions of linearisations of equation sets and are useful in a number of contexts. As mentioned above, they can be used to assess the validity of various equation sets as a function of scale ([Davies et al. \(2003\)](#)), and also to guide the choice of prognostic variables and vertical grid staggering ([Thuburn et al. \(2002a\)](#)). In [Staniforth et al. \(2004\)](#), spatially continuous normal modes are used to examine the stability properties of the [Davies et al. \(2004\)](#) dynamical core by solving a derived polynomial dispersion relation. Such an approach focusses on the stability characteristics of the time discretisation and has the virtue of simplicity, but neglects the impact of the spatial discretisation, including non-uniform resolution, and the application of boundary conditions. These may however be included using the framework of matrix stability analysis.

For a given temporal and spatial discretisation, the matrix stability analysis proceeds by first linearising the discretisations about a basic state, and then expressing the resulting set of linear difference equations as

$A\mathbf{x}^{n+1} = B\mathbf{x}^n$. Here the matrices A and B together define the vertical discretisation, $\mathbf{x}^n \equiv [u, v, w, \theta, \rho, \pi]^T$ is the transpose of the model's discrete state vector at time level n , and $u \equiv [u_1, u_2, \dots]^T$ is the vector of values of u at the set of discrete vertical levels $z = z_1, z_2, \dots$, with similar definitions for the other model variables v , w , θ , ρ and π . The generalised discrete eigenproblem $B\mathbf{x}^n = \lambda A\mathbf{x}^n$ is obtained by setting $\mathbf{x}^{n+1} \equiv \lambda \mathbf{x}^n$, and the discretisation is stable provided $|\lambda| \leq 1$.

In [Cordero et al. \(2002\)](#), this framework is used to assess the impact of using different implementations of vertical boundary conditions in a one-dimensional (column model) version of the [Davies et al. \(2004\)](#) dynamical core. In [Cordero et al. \(2004\)](#), the impact of (one- and two-term) extrapolated trajectory schemes on the stability properties of centred semi-implicit semi-Lagrangian schemes is similarly analysed. It is found that, in the absence of any controlling mechanism, both extrapolated trajectory schemes are unstable. Additionally they can significantly distort the vertical structure of the modes. Though not studied there, the analogous distortion of Rossby and gravity waves could be expected to be deleterious to a forecast model.

4 Physics-dynamics coupling

Physics parametrisation packages are key elements in the success of numerical weather and climate prediction models. The accuracy and complexity of these schemes continues to increase apace. Similarly, the accuracy of dynamical cores has continued to steadily improve. However, a chain is only as strong as its weakest link, e.g. two 2nd-order components coupled in a 1st-order manner imply a 1st-order model. The link coupling the physics package to the inviscid, adiabatic dynamical core has received little attention. It is therefore important for the continued improvement of models that the virtues and vices of the various strategies employed in such coupling are well understood, and that the vices are addressed.

In numerical models a distinction is usually made between fast and slow timescales because of differing stability considerations. An explicit time discretisation generally has the virtue of simplicity. For a slow timescale process, computational efficiency is usually not hindered by an associated stability-limited timestep and an $O(\Delta t)$ accurate discretisation therefore arguably suffices. However, for a fast timescale process, an explicit time-discretisation generally unduly limits the timestep due to an overly-restrictive stability condition. Therefore a more costly implicit time-discretisation is usually adopted. Even so, whilst this can address the stability issue, if the resulting discretisation is only $O(\Delta t)$ accurate, then the timestep may still be unduly limited due to time truncation error. This motivates an $O(\Delta t^2)$ -accurate implicit time discretisation of fast processes.

However, in a model there are several distinct processes (e.g. the dynamical core and each component of the physics package) each with their own timescale(s). The use of an implicit scheme to solve simultaneously for the time tendency of the complete model, though appealing, is currently prohibitively expensive, at least in an operational setting, and is likely to remain so for the foreseeable future. This is because of the expense of solving a modified Helmholtz problem which consists of contributions from both the dynamics *and* the physics package. The solution is to apply some form of splitting in which the time tendency due to the different elements of a model are evaluated separately, and then combined in some way to generate the complete model tendency. All operational models employ some form of splitting. The problem is that splitting in general introduces errors additional to the truncation errors associated with each individual process. With large timesteps, of the size permitted by semi-implicit semi-Lagrangian schemes, such errors can dominate the model error. The question is therefore: "How to determine the optimal way of performing such splitting?"

A methodology for analysing the numerical properties of such splitting schemes is developed in [Staniforth et al. \(2002a,b\)](#). A canonical problem is introduced to idealise both the dynamics (with terms to represent both fast and slow propagating modes), and the parametrisations of fast and slow, oscillatory and damped, physical processes. It permits the examination of a broad set of physics-dynamics coupling issues, whilst keeping the

analysis tractable. Any given coupling scheme can be assessed in terms of its numerical stability and of the accuracy of both its transient and steady-state responses.

For the reasons discussed above, fully implicit coupling is impracticable, as is fully explicit coupling due to timestep restrictions. A popular approach is “split-implicit” coupling in which a dynamics predictor is followed by a physics corrector. It addresses the stability issue of an explicit coupling whilst keeping the physics discretisation distinct from the dynamics discretisation. However, using the framework of Staniforth et al. (2002a,b), it is found that the steady-state solution is corrupted and the forced response can be spuriously amplified by an order-of-magnitude. This motivated the “symmetrised split-implicit” coupling in which two physics discretisations are arranged symmetrically around a dynamics sub-step. The analysis shows that this addresses the stability and accuracy deficiencies of an explicit coupling whilst still correctly representing the exact steady-state solution for constant forcing. It also keeps the physics discretisation distinct from the dynamics one. It partially shares the disadvantage of the fully implicit model inasmuch as the second physics sub-step is an implicit discretisation of the highly nonlinear physics. However the usual column-based physical parametrisations are such that the discrete set of nonlinear equations can be solved column-by-column, greatly reducing the computational cost.

This early work was done in the context of a physics package comprising only one component. In a typical model, however, there are at least four distinct components, each with different characteristics. The work of Dubal et al. (2004, 2005) therefore extends the above-described framework to examine the coupling of a mix of physical parametrisations of various damping and oscillatory processes associated with a range of timescales. Various coupling strategies have been examined but none has been found which performs uniformly well. Only rather general conclusions can be drawn. For example, there are two generic splitting schemes: sequential-splitting, in which the model’s tendency is updated sequentially using the tendency due to each physics component in turn; and parallel-splitting, in which the model’s tendency is updated simply by summing, independently, the tendencies of each physics component. It is found that sequential splitting is more flexible in its ability to eliminate splitting errors than parallel splitting. A disadvantage is that the sequential approach is sensitive to the order in which the physics components are applied. In practice a mix of sequential schemes for the fast timescale physics, and parallel schemes for the slow timescale ones, appears to optimise the overall coupling strategy. It is then found that the slower processes, such as radiation, should appear near the centre of the timestep, with the faster processes, such as boundary layer diffusion, coupled implicitly at the end of the timestep.

The framework of Staniforth et al. (2002a,b) can also be used to analyse the problem of spurious computational resonance in a semi-implicit semi-Lagrangian model. Traditionally, this has arisen in the presence of stationary spatial forcing, specifically that due to orography (Rivest et al. (1994)). In this case, spurious resonance is absent when a Courant number restriction on timestep is satisfied. Staniforth et al. (2002a) show that time-dependent forcing, such as that due to the physics package, can also give rise to spurious resonance. Importantly though, the Courant number limitation on the timestep is then twice as restrictive as that for stationary forcing, thereby exacerbating the problem of spurious computational resonance with long timesteps.

5 Conclusion

A unified approach to NWP and climate modelling for multiscale applications implies a strong constraint on the design of a dynamical core. The continuous system requires consideration of a suitable equation set and vertical coordinate. Semi-implicit SL methods are a proven approach for the discrete system. However further consideration still needs to be given to aspects such as: conservation; vertical discretisation; stability and accuracy of departure-point computation; and the coupling with physical parametrisations. Recent research at the Met Office on these dynamical core issues is reviewed herein.

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