Recent Research for Dynamical Cores of Nonhydrostatic, Deep-Atmosphere, Unified Models

Nigel Wood, Andrew Staniforth

and

The Dynamics Research Group

Met Office

Outline

- General considerations
- Key Ingredients
- Specific areas – a whistlestop tour
  - Continuous aspects
  - Discrete aspects
- Summary

The Met Office's Unified Model

Unified Model (UM) in that *single* model for:

- Operational forecasts at
  - Mesoscale (resolution approx. 10km)
  - Global scale (resolution approx. 50km)
- Global and regional climate predictions (resolution approx. 100km, run for 10-100 years)
- + Research mode (1km - 10m) and single column model

Operational Requirement

Current global model:

- Forecast to 6 days
- Time step = 20 minutes ⇒ 432 time steps
- Resolution = 432x325x38 = 5.3M grid points
- To run in 90 minute slot, including data assimilation and output

Design Requirements

- Highly efficient
- Yet robust (numerically stable) for both
  - weather forecasting
  - long term climate integrations
- Accurate for scales of interest
  - second-order or better
  - balance spatial and temporal truncation errors
- Conservative
  - ideally preserve all conservation properties
  - at best aim for important ones – mass (species), angular momentum, energy, PV
- Flexible
  - long term development path

Geometry

- Irregular surface ``removed'' via simple vertical co-ordinate transformation
- Atmosphere then spheroidal shell
- Simple geometry not afforded to
  - engineering flows
  - oceanography
- Capitalise and solve global system in spherical polar co-ordinates
- BUT - the pole problem!

Δx=900m $\Rightarrow$ CFL=(4/3)xpropagation speed!

Met Office

Modes of Response

- 1 Rossby (meteorological/slow) mode:
  - Synoptically most important
  - Inertia/Coriolis $\Rightarrow C_s \sim U$
- 2 Gravity modes:
  - Mesoscale/local interest
  - Inertia/buoyancy $\Rightarrow C_s \sim U \pm 50 ms^{-1}$ ($\sim U \pm 320 ms^{-1}$ external)
- 2 Acoustic modes:
  - Little meteorological interest
  - Inertia/compressibility $\Rightarrow C_s \sim U \pm 320 ms^{-1}$

Met Office

Challenge!

- How to stably discretise equations
- Whilst accurately capturing modes of interest
- In a finite time?
- Primarily a temporal discretisation problem
- Solution (Robert 1981, Staniforth& Côté MWR 1991) is to combine
  - semi-implicit and
  - semi-Lagrangian schemes
- But even then…

Key design ingredients

- Equation set
  - Form of equations/approximations etc
  - Choice of prognostic variables
  - Vertical coordinate
  - Impact on conservation issues
- Temporal discretization
  - Handle fast terms implicitly
  - Nonlinear terms
  - Helmholtz solver

- Spatial discretization
  - Choice of finite representation
  - Staggering in horizontal and vertical
  - Stretched grid: horizontal as well as vertical
  - Conservation properties
  - Boundary conditions
- Semi-Lagrangian aspects
  - Couples time and space
  - Impact on conservation
  - Trajectory calculations
- Coupling to Physics

Some specific topics

- Continuous Aspects
  - Equation set
  - Vertical coordinate
  - Energetics
- Discrete Aspects
  - Conservative Semi-Lagrangian advection
  - Semi-Lagrangian trajectories
    - » Accuracy – dynamical equivalence
    - » Stability – discrete normal mode analysis
- Coupling the dynamical core to the physics

(1) The Equation Set

- All models approximate the full equations
  - Specifically all make spherical-geopotential approximation
- *Almost* all models make the shallow atmosphere approximation
- *Almost* all operational global models make the hydrostatic approximation (filters horizontally propagating acoustic modes)
- Desirable (essential?) that approximated equation set is dynamically consistent in the sense (White et al) it:
  - Possesses conservation principles for energy, angular momentum and potential vorticity
  - Has a Lagrangian form of the momentum equation
  - White et al (2004) discuss 4 such models used operationally

Met Office

"Unapproximated" Equation Set

$$\frac{D_r u}{Dt} - \frac{uv\tan\phi}{r} - 2\Omega\sin\phi v + \frac{c_{pd}\theta_v}{r\cos\phi}\frac{\partial\Pi}{\partial\lambda} = \boxed{-\left(\frac{uw}{r} + 2\Omega\cos\phi w\right)} + S^u$$

$$\frac{D_r v}{Dt} + \frac{u^2\tan\phi}{r} + 2\Omega\sin\phi u + \frac{c_{pd}\theta}{r}\frac{\partial\Pi}{\partial\phi} = \boxed{-\left(\frac{vw}{r}\right)} + S^v$$

$$\left\{\frac{D_r w}{Dt}\right\} + c_{pd}\theta_v\frac{\partial\Pi}{\partial r} + \underbrace{\frac{\partial\Phi}{\partial r}}_{\approx\ g} = \boxed{\frac{\left(u^2+v^2\right)}{r} + 2\Omega\cos\phi u + S^w}$$

$$\frac{D_r}{Dt}\left(\rho_y r^2\cos\phi\right) + \rho_y r^2\cos\phi\left(\frac{\partial}{\partial\lambda}\left[\frac{u}{r\cos\phi}\right] + \frac{\partial}{\partial\phi}\left[\frac{v}{r}\right] + \frac{\partial w}{\partial r}\right) = 0$$

$$\frac{D_r\theta}{Dt} = S^\theta$$

Met Office

Equation Set Options

	Deep	Shallow ($r \rightarrow a$, neglect boxed terms)
Non-hydrostatic	Complete equations (Met Office from 2002)	Non-hydrostatic shallow (Eg Tanguay et al/GEM)
Hydrostatic (neglect Dw/Dt)	Quasi-hydrostatic (Met Office 1991-2002)	Hydrostatic primitive (Eg ECMWF)

[Staniforth 2001; White et al 2004]

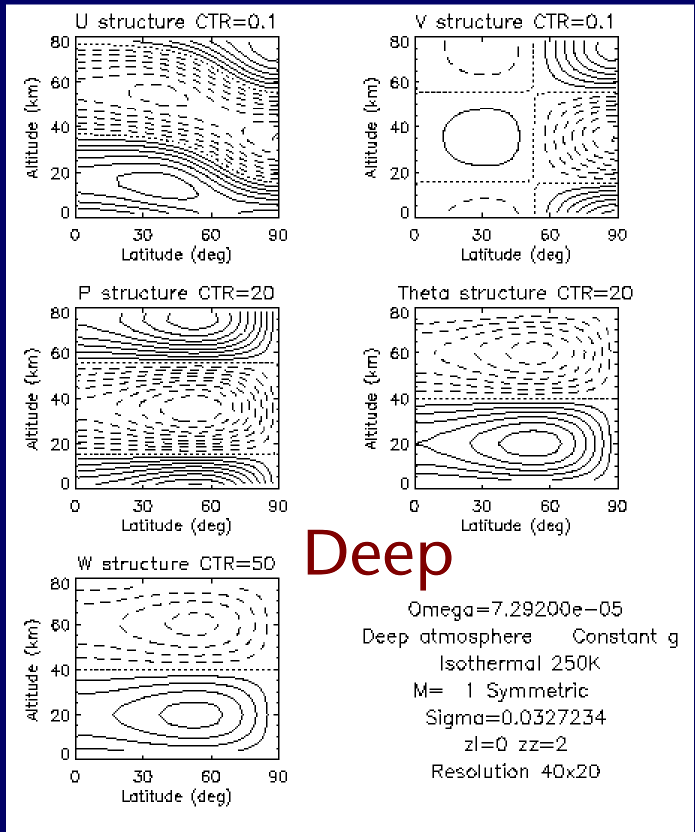
Met Office

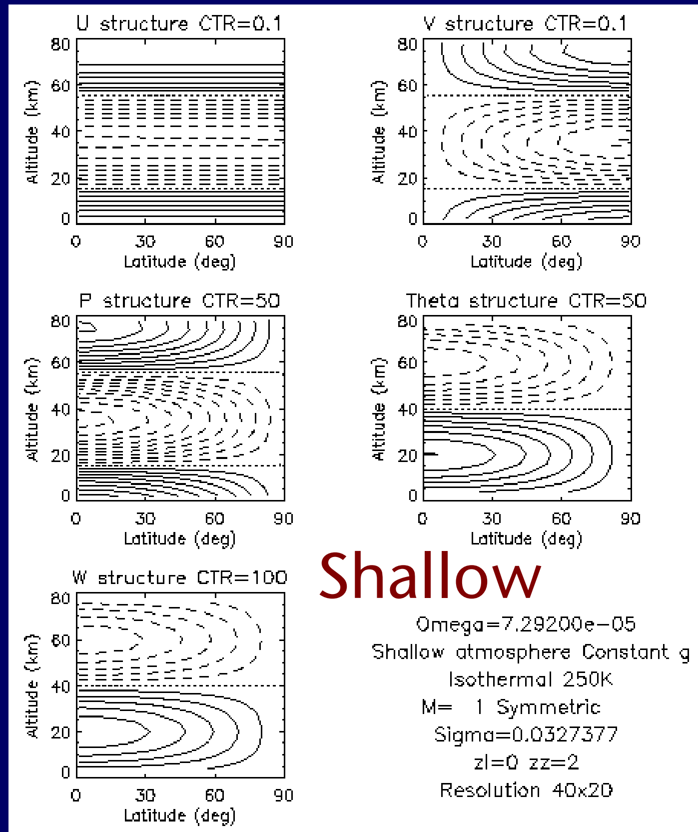
Normal mode analysis

- Normal mode analysis useful tool for studying fundamental properties of the equations
- Solutions of linearised, unforced equations
- Provide insight into impact of approximations to the equations
- Davies et al (2003) studied impact of hydrostatic and anelastic approximations
- Thuburn et al (2002) applied technique to investigate impact of gravity varying as $1/r^2$ vs. constant and of deep vs. shallow:

- Vertical variation of gravity ⇒ small (<1.5%) systematic decrease in frequency of normal modes
- Deep ⇒ nonzero w and θ perturbations for external Rossby and external acoustic modes (shallow = 0)
- No significant impact on spatial form of energetically important modes…
- …with only slight changes in frequency (1%)
- Significant changes in tropical structure of internal acoustic modes (relevant to forced case, eg tropical convection)

Normal mode structure: Deep vs Shallow





Latitude-height structure of longest meridional wavelength
2nd internal acoustic mode

[Thuburn et al 2002]

Using the unapproximated equations

- Met Office philosophy: use unapproximated equations; use numerics to do "filtering"
- Fully compressible, nonhydrostatic models do not filter the acoustic modes
  - Have to be handled implicitly if wish to avoid severe restriction on time step
- Deep atmosphere models have twice as many Coriolis terms to handle
  - Larger stencil if terms handled implicitly which stability requires for two-time-level scheme
- But, more accurate; more general (eg planetary atmospheres)

(2) Vertical Coordinate

- Hydrostatic models mostly use pressure as vertical coordinate
  - Simplifies equations (eq. of state diagnostic, density no longer appears in pressure gradient terms)
  - Reflects large scale dynamics of atmosphere
- Laprise (1992) defined a hydrostatic pressure (or equivalently mass) based coordinate
  - Plays the same role (and same advantages) in non-hydrostatic models but limited to shallow atmospheres
- Full equations = nonhydrostatic *and* deep (no shallow-atmosphere approximation)
- So pressure-based coordinate not an option (without approximation)?

- But! full equations in generalized vertical coordinates $\Rightarrow$ can define a mass-based coordinate (Π) for deep atmospheres with same properties as Laprise's shallow atmosphere hydrostatic pressure (π):

$$\frac{\partial \Pi}{\partial r} = -\rho \, g\Big|_{r=a} \left(\frac{r}{a} \right)^2$$

- Integrating this in height $\Rightarrow \Pi \propto$ mass of air in (diverging) column above given point
- Deep atmosphere distinguishes between mass and hydrostatic pressure viewpoints

[Wood and Staniforth 2003]

(3) Energetics (continuous)

- With this development, pressure-like coordinate possible
- Natural upper boundary = elastic upper lid
- What are the implications for energetics?
- In absence of forcing:

$$\frac{\partial}{\partial t}\int_V \rho E dV = -\int_{A_T} p_T \frac{\partial r_T}{\partial t} dA_T$$

- For a rigid lid (r_T=constant), as for Met Office height-based model,

$$\frac{\partial}{\partial t}\int_V \rho E dV = 0$$

- For elastic lid ($p_T = p_T(\lambda, \phi)$) independent of time)

$$\frac{\partial}{\partial t}\int_V \left(\rho E + p_T\right) dV = 0$$

- This is non-hydrostatic, non-shallow generalisation of Kasahara (1974)'s invariant:

$$\frac{\partial}{\partial t}\int_V \rho\left(K + c_p T + \Phi_H\right) dV = 0$$

- Also generalizes invariant energy forms of:
  - Laprise and Girard (1990) and Arakawa and Konor (1996) [hydrostatic, shallow]
  - Laprise (1992) [non-hydrostatic, shallow]

[Staniforth et al 2003]

(4) Semi-Lagrangian advection

- Discretize advective derivative as

$$\frac{D\phi}{Dt} \approx \frac{\phi^{t+\Delta t}(\boldsymbol{x}) - \phi^{t}(\boldsymbol{x} - \boldsymbol{U}\Delta t)}{\Delta t}$$

- $\boldsymbol{x}$ = arrival (grid) point
- $\boldsymbol{x} - \boldsymbol{U}\Delta t$ = departure point (solve $\frac{D\boldsymbol{x}}{Dt} = \boldsymbol{U}$)
- $\boldsymbol{U}\Delta t \equiv \boldsymbol{d}$ = upstream displacement vector
- No explicit stability constraints
- Two aspects to semi-Lagrangian schemes
  - 1) Evaluation of displacements/departure points
  - 2) Evaluation of function at departure point

[Staniforth and Côté 1991]

SL & Conservation

- Semi-Lagrangian schemes allow enhanced stability and accurate handling of meteorologically important slow mode
- Finite-difference interpolating form dissipative in nature (due to interpolation for second aspect)
- Two approaches to obtaining conserving forms:
  - *A posteriori* correction schemes (more or less ad hoc)
  - Finite-volume approach

  $\Rightarrow$ SLICE: Semi-Lagrangian Inherently Conserving and Efficient

- Two ingredients:
  - Rewrite Eulerian flux form

$$\frac{\partial \rho}{\partial t} + \nabla.\left(\rho \mathbf{u}\right) = 0$$

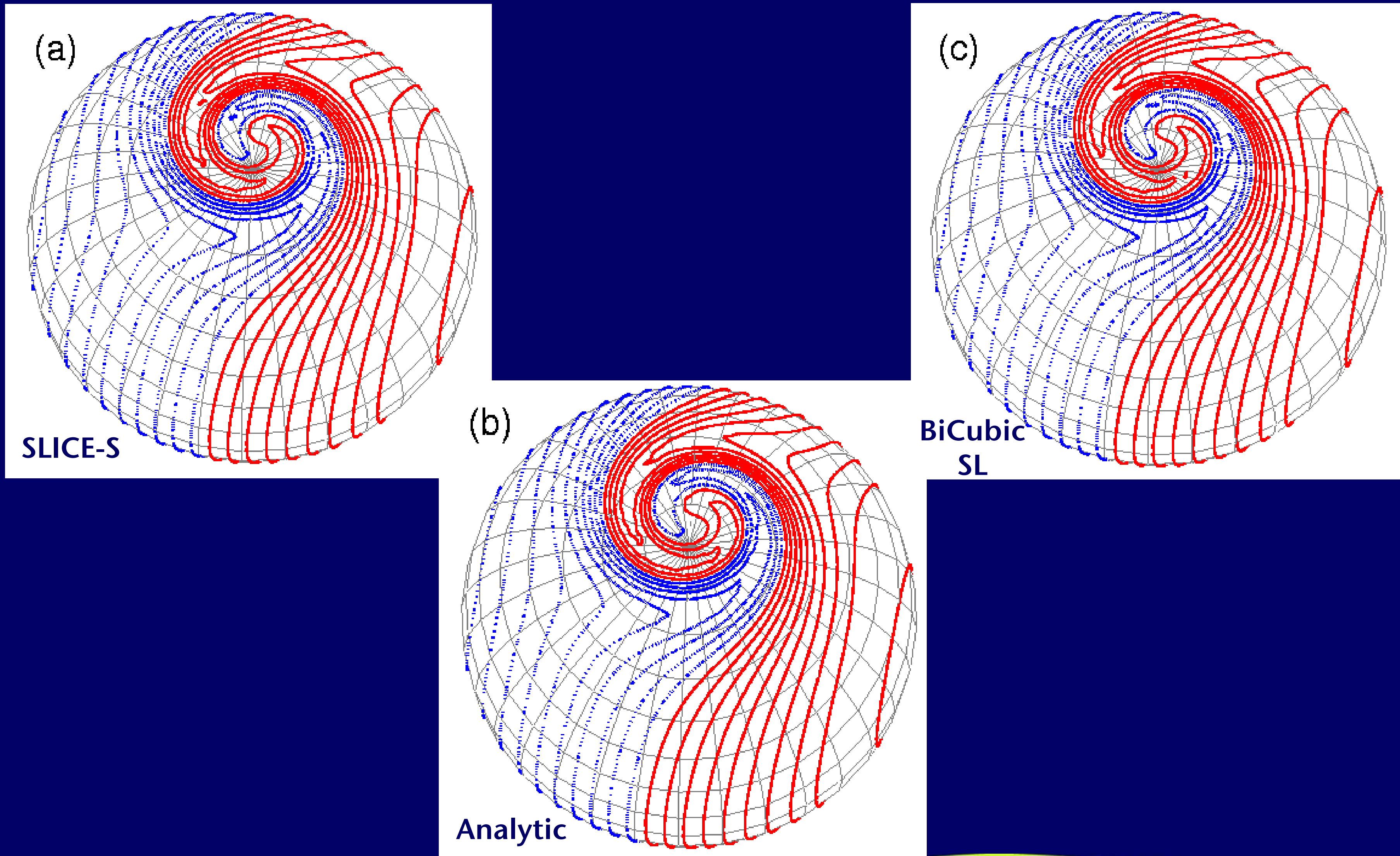
in finite-volume Lagrangian form

$$\frac{D}{Dt}\int_{\partial V} \rho dV = 0 \rightarrow M^{n+1} = M_d^n$$

  - Use Cascade remapping to enable split of 1 *n-dimensional* redistribution into n *one-dimensional* ones
    - » [Cascade interpolation preserves characteristics of flow and hence minimises splitting error.]

Met Office

Smooth Deformational Flow



[Zerroukat et al 2004]

(5) Trajectories: dynamical equivalence

- Key component of any semi-Lagrangian scheme is the calculation of the trajectories (displacement vector):

$$\frac{d\mathbf{r}}{dt} = \mathbf{u}(\mathbf{r}, t)$$

- This together with the momentum equation

$$\frac{d\mathbf{u}}{dt} = \mathbf{F}$$

$\Rightarrow$ angular momentum conservation

$$\frac{d\mathbf{r}\times\mathbf{u}}{dt} = \mathbf{G} \equiv \mathbf{r}\times\mathbf{F}$$

- Can a discrete form preserve this property?

- Discrete form requires estimate of trajectory mid-point velocity. Assume a form:

$$\mathbf{u}\left(t+\frac{\Delta t}{2}\right)=\alpha\mathbf{u}(t+\Delta t)+(1-\alpha)\mathbf{u}(t)$$

Then algebraic manipulation of discrete equations $\Rightarrow$

- Interpolation, α=1/2, preserves "dynamical equivalence"
- Not so for one-term, α=0, and two-term extrapolation:

$$\mathbf{u}\left(t+\frac{\Delta t}{2}\right)=\frac{3}{2}\mathbf{u}(t)-\frac{1}{2}\mathbf{u}(t-\Delta t)$$

- Will see later that interpolation scheme has other advantages

[White 2003; Staniforth et al 2003]

(6) Discrete Normal mode analysis

- Normal mode is fundamental solution of equation set
- Discrete normal modes characterize discretization:
  - Stability
  - Accuracy
- Linearize free equations

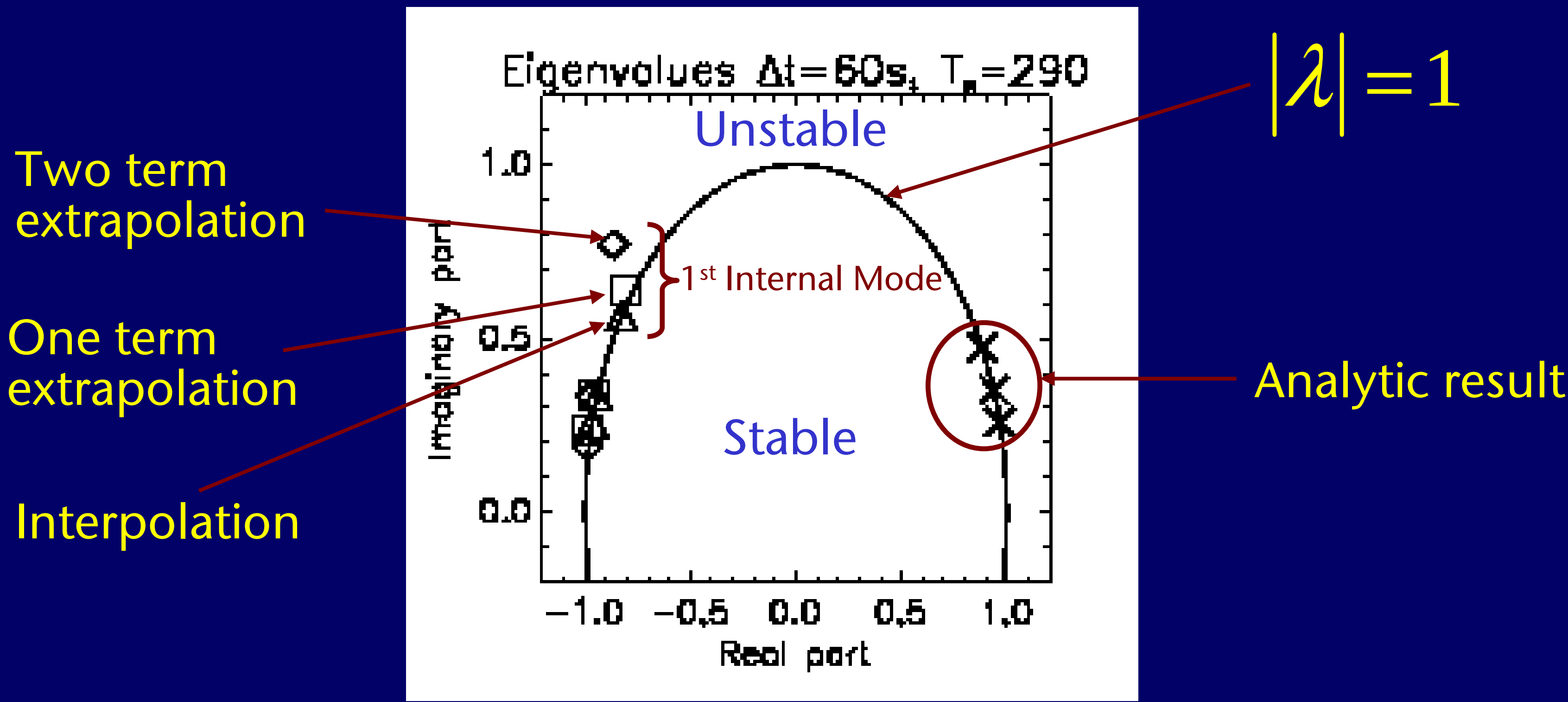
$$\mathbf{A}\mathbf{x}^{n+1} = \mathbf{B}\mathbf{x}^{n} \quad ; \quad \mathbf{x} \equiv \left[\mathbf{u}, \mathbf{v}, \mathbf{w}, \boldsymbol{\theta}, \boldsymbol{\rho}, \boldsymbol{\pi}\right]^{T}$$

- Eigen problem obtained by setting $\mathbf{x}^{n+1} = \lambda \mathbf{x}^{n}$

$$\mathbf{B}\mathbf{x}^{n} = \lambda \mathbf{A}\mathbf{x}^{n}$$

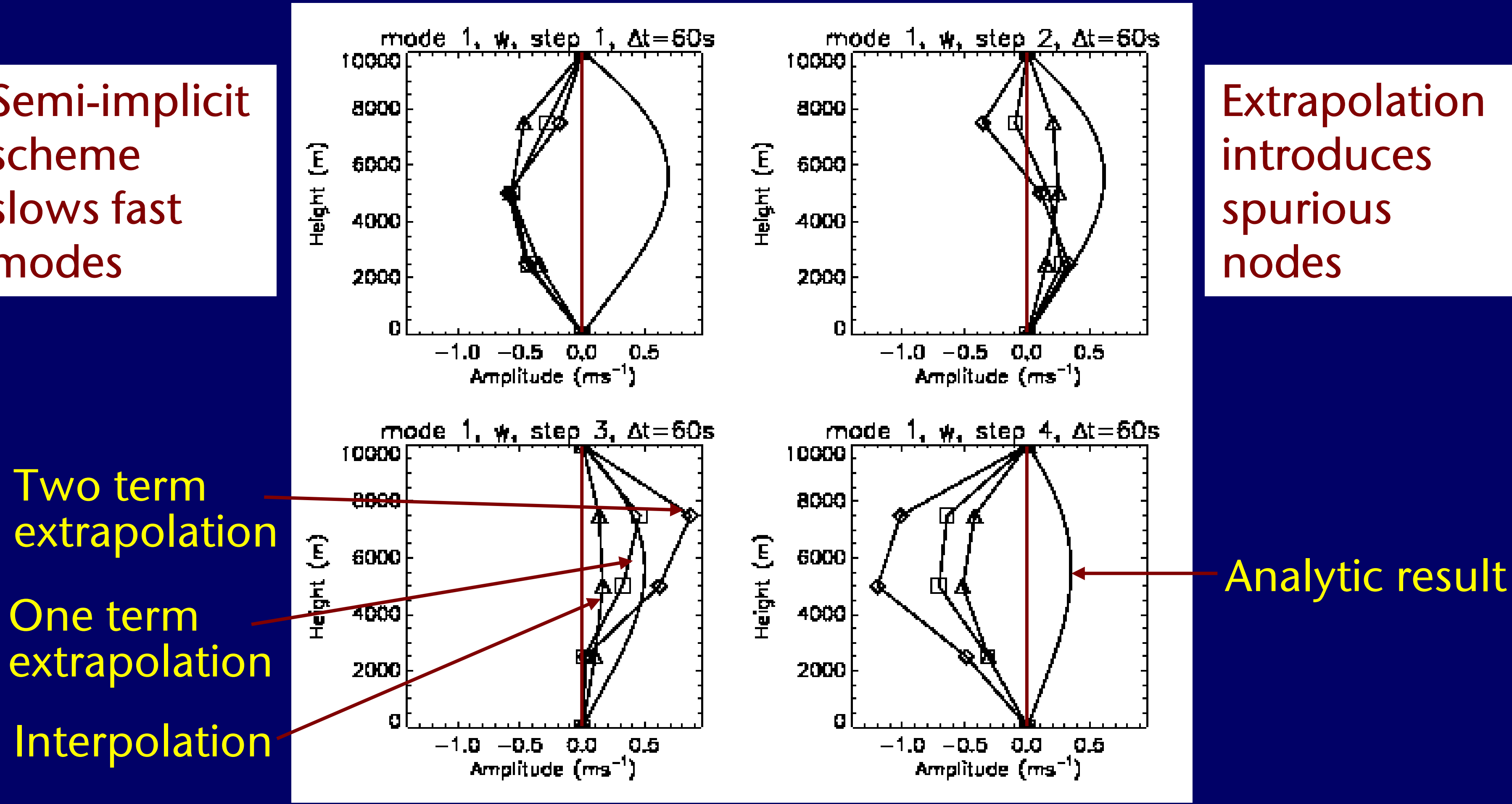
- Stable if $\left|\lambda\right| \leq 1$
- In general need to solve large matrix problem numerically

Impact of trajectory calculation on acoustic mode stability



Impact of trajectory calculation on acoustic mode structure

Semi-implicit scheme slows fast modes



Extrapolation introduces spurious nodes

[Cordero et al 2002 & 2004]

Met Office

(7) Physics-Dynamics Coupling

- Weakest link? Two 2nd order schemes coupled in 1st order way ⇒ 1st order model
- Aim is to provide simple framework in which to investigate numerics of coupling scheme:
  - Stability
  - Accuracy
  - Spurious Resonance
  - Steady-state (slow mode)
- Dynamics + 1 Physics (basic method inc. advection)
- 2 Physics (sequential vs parallel)
- Multiple physics (mixed sequential/parallel)

Met Office

Model problem with multiple time-scales

- One slow (time-scale » time-step) & one fast (time-scale $\lesssim$ time-step) process:

$$\varepsilon \frac{dF(t)}{dt} = e^{it} - \sigma F(t), \quad \varepsilon << 1, \quad \sigma = \beta + i\alpha$$

- Apply Symmetrized Sequential-Split method:

$$\frac{F^{*} - F(t)}{\Delta t} = \frac{\eta}{\varepsilon}\left[\xi_2 e^{i(t+\Delta t)} + (1-\xi_2)e^{it}\right]$$

$$\frac{F^{**} - F^{*}}{\Delta t} = -\frac{\sigma}{\varepsilon}\left[\xi_1 F^{**} + (1-\xi_1)F^{*}\right]$$

$$\frac{F(t+\Delta t) - F^{**}}{\Delta t} = \frac{(1-\eta)}{\varepsilon}\left[\xi_3 e^{i(t+\Delta t)} + (1-\xi_3)e^{it}\right]$$

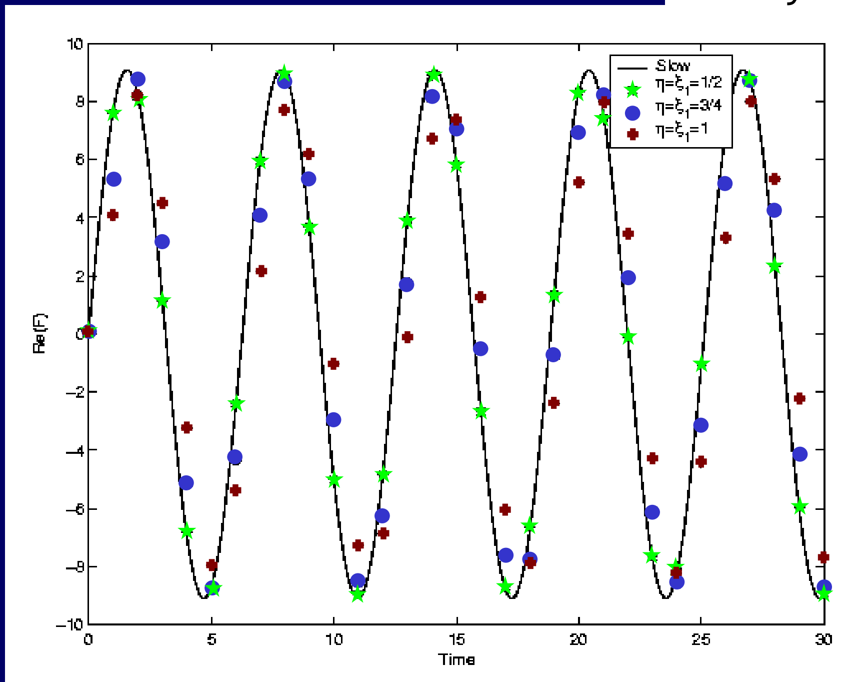
[Staniforth et al 2002a&b]

Example: Timestep sensitivity

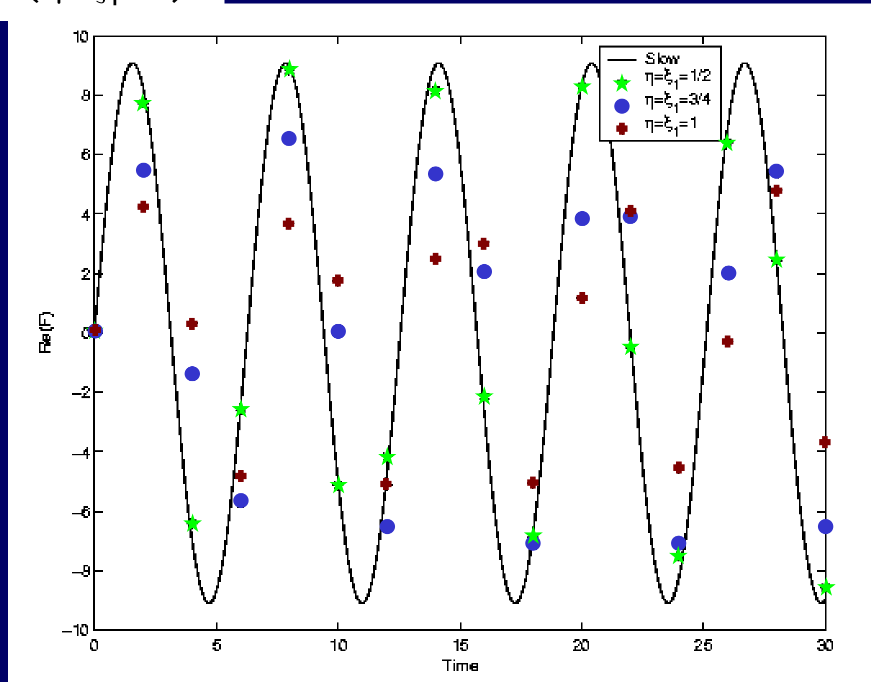
Two timescales: fast (2π/10 s) and slow (2π s)

- ★ 2nd order ($\eta=\xi_1=1/2$)
- ● Off-centred ($\eta=\xi_1=3/4$)
- ● Fully implicit ($\eta=\xi_1=1$)

$\Delta t=1$ s



$\Delta t=2$ s



[Dubal et al 2004]

Summary

- Global Unified Modelling approach implies strong constraint on model design
- Continuous system requires consideration of:
  - Equation set
  - Vertical coordinate
  - Energetics
- Discrete system:
  - Semi-Lagrangian semi-implicit proven approach
- But consideration still needs to be given to:
  - Conservation
  - Vertical discretization (eg Untch & Hortal 2004)
  - Stability and accuracy of departure point calculations
  - Coupling of dynamics to physics