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The shallow water limit of the Zakharov Equation and consequences for (freak) wave prediction

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Abstract

Finite amplitude deep-water waves are subject to nonlinear focussing, which when the phases are right, may give rise to giant waves or freak waves. The same process is responsible for the Benjamin-Feir instability. In shallow water, finite amplitude surface-gravity waves generate a current and deviations from the mean surface elevation. This stabilizes the Benjamin-Feir instability and the process of nonlinear focussing ceases to exist when $kh < 1.363$. This is a well-known property of surface gravity waves (Benjamin, 1967; Whitham, 1974) and, here it is shown for the first time, that the usual starting point for wave evolution studies, namely the Zakharov equation, shares this property as well.

Consequences for (freak) wave prediction are pointed out.

1 Introduction.

Since the beginning of the 1990's we have seen a rapid increase in our understanding of the generation of large freak waves on the open ocean. Nowadays the consensus is that third order nonlinear interactions enhance freak wave appearance and are the primary cause for the generation of freak waves. Exceptions are cases of strong wave-current interaction or wave diffraction behind islands.

In deep water third-order interactions lead to focussing of wave energy. However, most observations are at locations close to the coast, where shallow water effects may become important. For example, the famous Draupner freak wave was observed in water with a depth h of 69 m, and using the observed freak frequency and the shallow water dispersion relation one infers that the dimensionless depth k_0h is just above 1, $k_0h = 1.2$.

This therefore prompted a study into the effects of shallow water on the generation of extreme waves, with some unexpected consequences. In shallow water finite amplitude waves generate a wave-induced current, hence for decreasing depth less and less wave energy is available for nonlinear focussing. As a consequence the process of nonlinear focussing ceases to exist for sufficiently small water depth, $k_0h < 1.363$. This well-known result was first found by Benjamin (1967) and Whitham (1974) when studying the instability of a uniform, finite amplitude wave train. The consequence of this result should be clear: in shallow water it is less likely that giant freak waves occur.

Before we do this, we first establish that the basic evolution equation for surface gravity waves, the Zakharov equation, indeed correctly accounts for the stabilizing effects of wave-induced current and mean sea surface elevation. This holds for intermediate water depth and even for very shallow water when the dynamics of the waves is determined by the Korteweg-de Vries equation.

An important implication for spectral wave modelling in shallow waters is that around $k_0h = 1.363$ there is a considerable reduction of the nonlinear transfer rates. This will be shown explicity in this work by means of results of Monte Carlo forecasting of the Zakharov Equation. In addition, these results support the conjecture that freak waves indeed occur less frequently in waters of intermediate depth.

These results are unexpected because the 'classical' approach (Herterich and Hasselmann, 1980) predicts a vanishing of the nonlinear transfer at much smaller values of dimensionless depth, $k_0h \simeq 0.6 - 0.7$. This is an important difference because in shallow water a 'typical' saturated windsea corresponds to a dimensionless depth k_0h of about 1. In that case, in a considerable part of the wave spectrum the balance is determined by wind input and dissipation only.

2 Preliminaries.

The formulae for the second and third order interaction coefficients follow from Krasitskii (1994), while the expression for T in the Zakharov equation is obtained from Zakharov (1992).

Introduce

$$T_0 = \tanh kh_0 \tag{1}$$

with k the wavenumber and h_0 the mean depth. Then the dispersion relation reads

$$\omega^2 = gkT_0. \tag{2}$$

In addition introduce

$$q = \omega^2/g. \tag{3}$$

In shallow water we have the following relation between $\hat{\eta}$, $\hat{\psi}$ and the action density variable $A(\vec{k},t)$

$$\hat{\eta} = \sqrt{\frac{\omega}{2g}}\left(A(\vec{k}) + A^*(-\vec{k})\right), \hat{\psi} = -i\sqrt{\frac{g}{2\omega}}\left(A(\vec{k}) - A^*(-\vec{k})\right). \tag{4}$$

For a homogeneous random sea one then finds the following relation between the action density spectrum N and the surface elevation spectrum F_η,

$$N = \frac{gF_\eta}{\omega}. \tag{5}$$

Zakharov (1968) found that gravity waves are a hamiltonian system where the hamiltonian is the energy E, the sum of potential and kinetic energy. In terms of the action variable $A(\vec{k},t)$ he obtained the following expansion

$$\begin{aligned} E = &\int d\vec{k}_1 \omega_1 A_1 A_1^* + \int d\vec{k}_{1,2,3}\delta_{1-2-3}V^{(-)}_{1,2,3}\left[A_1^*A_2A_3 + c.c.\right] + \frac{1}{3}\int d\vec{k}_{1,2,3}\delta_{1+2+3}V^{(+)}_{1,2,3}\left[A_1A_2A_3 + c.c.\right] \\ &+ \int d\vec{k}_{1,2,3,4}\delta_{1-2-3-4}W^{(1)}_{1,2,3,4}\left[A_1^*A_2A_3A_4 + c.c.\right] + \frac{1}{2}\int d\vec{k}_{1,2,3,4}\delta_{1+2-3-4}W^{(2)}_{1,2,3,4}A_1^*A_2^*A_3A_4 \\ &+ \frac{1}{4}\int d\vec{k}_{1,2,3,4}\delta_{1+2+3+4}W^{(4)}_{1,2,3,4}\left[A_1^*A_2^*A_3^*A_4^* + c.c\right] \end{aligned} \tag{6}$$

The second-order coefficients become

$$\begin{aligned} V^{(\pm)}_{1,2,3} = \frac{1}{4\sqrt{2}}\Bigg\{&\left[\vec{k}_1.\vec{k}_2 \pm q_1q_2\right]\left(\frac{g\omega_3}{\omega_1\omega_2}\right)^{1/2} + \\ &\left[\vec{k}_1.\vec{k}_3 \pm q_1q_3\right]\left(\frac{g\omega_2}{\omega_1\omega_3}\right)^{1/2} + \left[\vec{k}_2.\vec{k}_3 + q_2q_3\right]\left(\frac{g\omega_1}{\omega_2\omega_3}\right)^{1/2}\Bigg\} \end{aligned} \tag{7}$$

with $k_i = |\vec{k}_i|, \omega_i = \omega(k_i)$. The relevant third-order coefficient is $W = W^{(2)}_{1,2,3,4}$. It becomes

$$W_{1,2,3,4} = U_{-1,-2,3,4} + U_{3,4,-1,-2} - U_{3,-2,-1,4} - U_{-1,3,-2,4} - U_{-1,4,3,-2} - U_{4,-2,3,-1} \tag{8}$$

with

$$U_{1,2,3,4} = \frac{1}{16}\left(\frac{\omega_3\omega_4}{\omega_1\omega_2}\right)^{1/2}\left[2(k_1^2q_2 + k_2^2q_1) - q_1q_2\left(q_{1+3} + q_{2+3} + q_{1+4} + q_{2+4}\right)\right]. \tag{9}$$

The hamiltonian is in terms of the action variable A. It contains cubic terms which are non-resonant and which may be eliminated by means of a canonical transformation

$$A_1 = a_1 + \int d\vec{k}_{2,3} \left\{ A^{(1)}_{1,2,3} a_2 a_3 \delta_{1-2-3} + A^{(2)}_{1,2,3} a_2^* a_3 \delta_{1+2-3} + A^{(3)}_{1,2,3} a_2^* a_3^* \delta_{1+2+3} \right\} + \mathscr{O}(a^3) \quad (10)$$

where

$$\begin{aligned} A^{(1)}_{1,2,3} &= -\frac{V^{(-)}_{1,2,3}}{\omega_1 - \omega_2 - \omega_3} \\ A^{(2)}_{1,2,3} &= -2\frac{V^{(-)}_{2,1,3}}{\omega_1 + \omega_2 - \omega_3} \\ A^{(3)}_{1,2,3} &= -\frac{V^{(+)}_{1,2,3}}{\omega_1 + \omega_2 + \omega_3} \end{aligned} \quad (11)$$

The resulting energy density then becomes

$$E = \int d\vec{k}_1 \omega_1 a_1^* a_1 + \frac{1}{2} \int d\vec{k}_{1,2,3,4} T_{1,2,3,4} a_1^* a_2^* a_3 a_4 \delta_{1+2-3-4}, \quad (12)$$

where the nonlinear interactions coefficient $T_{1,2,3,4}$ reads

$$\begin{aligned} T_{1,2,3,4} = W_{1,2,3,4} & \\ & - V^{(-)}_{1,3,1-3} V^{(-)}_{4,2,4-2} \left[\frac{1}{\omega_3 + \omega_{1-3} - \omega_1} + \frac{1}{\omega_2 + \omega_{4-2} - \omega_4} \right] \\ & - V^{(-)}_{2,3,2-3} V^{(-)}_{4,1,4-1} \left[\frac{1}{\omega_3 + \omega_{2-3} - \omega_2} + \frac{1}{\omega_1 + \omega_{4-1} - \omega_4} \right] \\ & - V^{(-)}_{1,4,1-4} V^{(-)}_{3,2,3-2} \left[\frac{1}{\omega_4 + \omega_{1-4} - \omega_1} + \frac{1}{\omega_2 + \omega_{3-2} - \omega_3} \right] \\ & - V^{(-)}_{2,4,2-4} V^{(-)}_{3,1,3-1} \left[\frac{1}{\omega_4 + \omega_{2-4} - \omega_2} + \frac{1}{\omega_1 + \omega_{3-1} - \omega_3} \right] \\ & - V^{(-)}_{1+2,1,2} V^{(-)}_{3+4,3,4} \left[\frac{1}{\omega_{1+2} - \omega_1 - \omega_2} + \frac{1}{\omega_{3+4} - \omega_3 - \omega_4} \right] \\ & - V^{(+)}_{-1-2,1,2} V^{(+)}_{-3-4,3,4} \left[\frac{1}{\omega_{1+2} + \omega_1 + \omega_2} + \frac{1}{\omega_{3+4} + \omega_3 + \omega_4} \right]. \end{aligned} \quad (13)$$

3 Comparison with known results

The Zakharov equation now follows from Hamiltons equations $\partial a/\partial t = -i\delta E/\delta a^*$, or,

$$\frac{\partial a_1}{\partial t} + i\omega_1 a_1 = -i \int d\vec{k}_{2,3,4} T_{1,2,3,4} a_2^* a_3 a_4 \delta_{1+2-3-4}, \quad (14)$$

This very compactly written equation contains a lot of interesting physics, and here we would like to explore this for the general case of intermediate depth. We will try to derive for the case of a single wave important relations such as the dispersion relation, the expression for the mean surface elevation and the mean current, and we will try to compare with known results from Davey-Stewartson (1974) and Whitham (1974). In particular, we would like to check that wave-induced current and mean surface elevation indeed have a damping effect on the Benjamin-Feir instability in such a way that for $k_0 h_0 = 1.363$ the instability disappears. In other words, for $k_0 h_0 > 1.363$ nonlinearity focusses wave energy, while in the opposite case we have defocussing.

3.1 Summary of known results

In shallow water the wave-induced current and mean surface elevation have a stabilizing effect in such a way that for $k_0h_0 < 1.363$ the Benjamin-Feir Instability disappears and there is no focussing (Benjamin, 1967; Whitham, 1974). This is understood most easily from Whitham's variational approach. The nonlinear dispersion relation on a current β is found to be

$$\frac{(\omega-k\beta)^2}{gk\tanh kh} = 1+\frac{9T_0^4-10T_0^2+9}{4T_0^4}\,\frac{k^2E}{g}, \tag{15}$$

with $E = ga^2/2$ the wave energy for a single wave train with amplitude a. This dispersion relation is accompanied by equations for the current β and mean elevation $b = h - h_0$. Whitham finds

$$b = -\frac{h_0}{c_S^2 - v_g^2}\,\frac{S}{h_0}, \tag{16}$$

with $c_S^2 = gh_0$, $\vec{v}_g = \partial\omega/\partial\vec{k}$ and S the radiation stress,

$$S = \left(\frac{2v_g}{c_0} - \frac{1}{2}\right)E, \tag{17}$$

while

$$U = \beta + \frac{E}{c_0h_0} = \frac{v_g}{h_0}\,b \tag{18}$$

Linearizing in b, the dispersion relation becomes

$$\omega = \omega_0 + \Omega_2(k)\frac{k^2E}{c_0}, \omega_0^2 = gkT_0, \tag{19}$$

where

$$\Omega_2(k) = \frac{9T_0^4-10T_0^2+9}{8T_0^3} - \frac{1}{kh_0}\left\{\frac{\left(2v_g - c_0/2\right)^2}{c_S^2 - v_g^2} + 1\right\}. \tag{20}$$

The stability of a uniform wave train is determined by the sign of the product of the second derivative of ω_0, denoted by ω_0'', and Ω_2. There is instability when $\omega_0''\Omega_2 < 0$ (Whitham, 1974). Now $\omega_0 = (gkT_0)^{1/2}$ and ω_0'' is always negative. The stability of a uniform surface gravity wave train is therefore determined by the sign of the nonlinear term: there is stability for negative Ω_2 and instability in the opposite case.

For large depth, the wave-induced current contribution vanishes, while $T_0 \rightarrow 1$. In that event, $\Omega_2(k) \rightarrow 1$ which results in the well-known nonlinear dispersion relation for deep-water waves. Clearly, as Ω_2 is positive, a deep-water uniform wave train is unstable, and the nonlinearity leads to focussing of wave energy. For shallow waters, the curly bracketed term in Eq. (20) becomes important. It is positive definite and leads to stabilization of the Benjamin-Feir instability. At $kh_0 = 1.363$, Ω_2 vanishes. Hence, for $kh_0 < 1.363$ a uniform wave train is stable as Ω_2 is negative.

In the opposite case of very small depth, hence $kh_0 << 1$, one finds that $\Omega_2 = -9/8(kh_0)^{-3}$ hence a uniform wave train is, as expected, stable. The resulting dispersion relation corresponds exactly with the nonlinear dispersion relation as obtained from the Korteweg-de Vries equation (Hasimoto and Ono, 1972).

3.2 Narrow-band approximation

We use the canonical transformation (10) up to second order in amplitude and as a starting point for the iteration we take

$$a_i = \hat{a}_i \delta(\vec{k}_i - \vec{k}_0), \; \hat{a}_i = \rho e^{-i\omega_i t}. \tag{21}$$

This gives for the action variable A to second order

$$A_1 = \hat{a}_i \delta(\vec{k}_1 - \vec{k}_0) - \left[\frac{V^{(-)}_{1,0,0}}{\omega_1 - 2\omega_0} \hat{a}_0^2 \delta(\vec{k}_i - 2\vec{k}_0) + \frac{V^{(+)}_{1,0,0}}{\omega_1 + 2\omega_0} \hat{a}_0^{*2} \delta(\vec{k}_i + 2\vec{k}_0) \right.$$
$$\left. +2 \int d\vec{k}_{2,3} \frac{V^{(-)}_{2,1,3}}{\omega_1 + \omega_2 - \omega_3} |\hat{a}_0|^2 \delta(\vec{k}_2 - \vec{k}_0) \delta(\vec{k}_3 - \vec{k}_0) \delta(\vec{k}_1 + \vec{k}_2 - \vec{k}_3) \right] \tag{22}$$

hence the second-order term generates a second harmonic contribution and a mean response (which for obvious reasons we have not yet written explicitely).

The second harmonics are easy to deal with because the denominators remain finite. The relevant matrix elements become

$$V^{(\pm)}_{1,0,0} = \frac{1}{4\sqrt{2}} \left\{ 2(\vec{k}_1 \cdot \vec{k}_0 \pm q_1 q_0) \left(\frac{g}{\omega_1} \right)^{1/2} + \frac{k_0^2 + q_0^2}{\omega_0} \sqrt{g\omega_1} \right\}, \text{ at } \vec{k}_1 = 2\vec{k}_0. \tag{23}$$

The mean flow contribution is much more awkward because of the singularity caused by the factor $\omega_1 + \omega_2 - \omega_3 = 0$ for $\vec{k}_2 \to \vec{k}_0$, $\vec{k}_3 \to \vec{k}_0$ and consequently $\vec{k}_1 \to 0$. Strictly speaking the mean response in the action density diverges and only the mean surface elevation remains finite. In addition, one obtains different answers depending on how the limits are taken. An example of a limit is the one where $\vec{k}_2 = \vec{k}_0$, and $\vec{k}_3 = \vec{k}_0$ while the limit $\vec{k}_1 \to 0$ is only taken afterwards. The resulting expression for the mean surface elevation is identical to the one given by Benjamin (1967). The problem with this limit is, however, that by choosing finite $\vec{k}_1$ one moves from the resonance surface determined by the resonance condition $\vec{k}_1 + \vec{k}_2 - \vec{k}_3 = \vec{0}$.

We prefer to stick to the resonance surface and to choose $\vec{k}_2$ and $\vec{k}_3$ slightly different in order to satisfy the resonance condition (see also, Gorman, 2003). Specifically, write

$$\vec{k}_3 = \vec{k}_0 + \vec{\varepsilon}, \; \vec{k}_2 = \vec{k}_0 \tag{24}$$

where ε is assumed to be small. Because of the resonance condition the wave length of wave '1' becomes very long, or,

$$\vec{k}_1 = \vec{\varepsilon}. \tag{25}$$

As a consequence, on the resonance surface the factor $\omega_1 + \omega_2 - \omega_3$ becomes equal to $k_1 c_S - \vec{k}_1 v_g$, with c_S the shallow water speed and v_g the group velocity.

Hence by choosing the wavenumbers in this fashion we are considering the mean surface elevation and the nonlinear transfer in the limit of a very long wave group! The mean flow response then becomes to lowest significant order

$$\langle A_1 \rangle = -B_0 |\hat{a}_0|^2 \delta(\vec{k}_1 - \vec{\varepsilon}), \tag{26}$$

where

$$B_0 = \frac{2V^{(-)}_{0,1,0}}{k_1 c_S - \vec{k}_1 . \vec{v}_g}, \tag{27}$$

and

$$V^{(-)}_{0,1,0} = \frac{1}{4\sqrt{2}} \left\{ \frac{k_0^2 - q_0^2}{\omega_0} \sqrt{g\omega_1} + 2\vec{k}_0 \cdot \vec{k}_1 \left(\frac{g}{\omega_1} \right)^{1/2} \right\}, \text{ for } \vec{k}_1 \to \vec{0}. \tag{28}$$

From now on consider only the one-dimensional case, nevertheless we will still use vector notation in order to distinguish magnitude of a vector from the usual vector (which in 1D carries a sign). Hence, write the action density as

$$A_1 = \hat{a}_1 \delta(\vec{k}_1 - \vec{k}_0) - B_2 \hat{a}_0^2 \delta(\vec{k}_1 - 2\vec{k}_0) - B_{-2} \hat{a}^{*2} \delta(\vec{k}_1 + 2\vec{k}_0) - B_0 |\hat{a}_0|^2 \delta(\vec{k}_1 - \vec{\varepsilon}) \tag{29}$$

where B_0 is given by Eq. (27) while

$$B_2 = \frac{V^{(-)}_{1,0,0}}{\omega_1 - 2\omega_0}, \text{ and } B_{-2} = \frac{V^{(+)}_{1,0,0}}{\omega_1 + 2\omega_0}. \tag{30}$$

The surface elevation now becomes

$$\eta = \int \mathrm{d}\vec{k}\, \hat{\eta}\, e^{i\vec{k}.\vec{x}} = \int \mathrm{d}\vec{k} \left(\frac{\omega}{2g} \right)^{1/2} A(\vec{k}) e^{i\vec{k}.\vec{x}} + \text{c.c.} \tag{31}$$

Substitution of (30) into (31) and introduction of the amplitude $a = (2\omega_0/g)^{1/2}\, \rho$ gives

$$\begin{aligned} \eta = a\cos\theta - \frac{a^2}{2\omega_0} \left(\frac{g\omega(\varepsilon)}{2} \right)^{1/2} \left[B_0(+\vec{\varepsilon}) + B_0(-\vec{\varepsilon}) \right] \\ - \frac{a^2}{\omega_0} \left(\frac{g\omega(2\vec{k}_0)}{2} \right)^{1/2} \left[B_2(2\vec{k}_0) + B_{-2}(-2\vec{k}_0) \right] \cos 2\theta, \end{aligned} \tag{32}$$

where $\theta = \vec{k}_0.\vec{x} - \omega_0 t$.

Using the expression for B_0, B_2 and B_{-2} and taking the limit of vanishing ε one finds explicitely

$$\eta = k_0 a^2 \Delta + a\cos\theta + k_0 a^2 P \cos 2\theta, \tag{33}$$

with

$$P = \frac{1}{4T_0} \left(\frac{3}{T_0^2} - 1 \right), \; T_0 = \tanh k_0 h_0,$$

while

$$\Delta = -\frac{1}{4} \frac{c_S^2}{c_S^2 - v_g^2} \left(\frac{2(1 - T_0^2)}{T_0} + \frac{1}{k_0 h_0} \right).$$

Both expressions agree with Whitham (1974, 13.123&16.99). A remarkable property of the mean flow response is that it does not vanish exponentially for large $k_0 h_0$. In stead it only vanishes like $1/k_0 h_0$!

In the Appendix a similar calculation is performed for the mean flow according to the Zakharov equation. To this end Whitham introduced in a natural way the wave-induced mass transport velocity u_w. With wave variance $E = 1/2ga^2$ one has,

$$u_w = \frac{E}{c_0 h_0} = \frac{1}{2} \frac{g k_0}{\omega_0 h_0} a^2 \tag{34}$$

and Whitham (1974) finds that the average mass transport velocity, defined as the sum of the wave-induced transport and the mean circulation velocity $\beta = \partial\langle\phi\rangle/\partial x$, or, $U = \beta + u_w$, obeys the relation (18), i.e. $U = v_g b/h_0$, where $b = k_0 a^2 \Delta$ is the mean surface elevation. We have determined the mean of the velocity potential, $\langle\phi\rangle$, from Zakharov's Hamiltonian approach in the Appendix, and it is found that Eq. (18) is indeed satisfied.

In order to obtain the dispersion relation for a single wave in shallow water we require $T_{0,0,0,0}$. Inspecting the general expression for T in Eq. (13) it is evident that once more the limit of equal wavenumbers is akward, since the first four terms show apparent singularities. We will treat this limit in a similar fashion as in the case of the mean surface elevation. The task is, however, simplified by the ubundance of symmetries of T. Let us start with the first singular term, and we perturb all wave numbers slightly, respecting the resonance condition in the Zakharov equation, hence

$$\vec{k}_1 = \vec{k}_0 + \vec{\varepsilon}_1, \vec{k}_2 = \vec{k}_0 + \vec{\varepsilon}_2, \vec{k}_3 = \vec{k}_0 + \vec{\varepsilon}_3, \vec{k}_4 = \vec{k}_0 + \vec{\varepsilon}_4, \tag{35}$$

in such a way that $\vec{\varepsilon}_1 + \vec{\varepsilon}_2 = \vec{\varepsilon}_3 + \vec{\varepsilon}_4$. The first bracketed term in (13), denoted by $f(\vec{d})$, becomes in lowest significant order

$$f(\vec{d}) = -\frac{|\vec{d}|}{16(|\vec{d}|c_S - \vec{d}\cdot\vec{v}_g)} \left\{ k_0^2(1-T_0^2)\frac{gc_S}{\omega_0} + \frac{2\vec{k}_0\cdot\vec{d}}{|\vec{d}|}\left(\frac{g}{c_S}\right)^{1/2} \right\}^2, \tag{36}$$

where $\vec{d} = \vec{\varepsilon}_1 - \vec{\varepsilon}_3$. The last singular term in (13) can be obtained from the first one by interchanging the indices 1, 3 and 4, 2. As a result, this last term equals $f(-\vec{d})$. Combining the two terms we have that their sum equals $f(\vec{d}) + f(-\vec{d})$ and is therefore independent of the sign of the difference vector $\vec{d}$.

The second and third singular term give a similar contribution and upon taking the limit of vanishing distance $\vec{d}$ one finds that the singular terms amount to

$$-\frac{1}{4}\frac{k_0^3 c_S^2}{c_S^2 - v_g^2}\left[\frac{\left(1-T_0^2\right)^2}{T_0} + \frac{4gv_g}{\omega_0 c_S^2}\left(1-T_0^2\right) + \frac{4}{k_0 h_0}\right] \tag{37}$$

Making use of the dispersion relation and the expression for the group speed Eq. (37) becomes

$$-\frac{k_0^3}{k_0 h_0}\left\{\frac{\left(2v_g - c_0/2\right)^2}{c_S^2 - v_g^2} + 1\right\}, \tag{38}$$

where $c_0 = \omega_0/k_0$ is the linear phase speed.

The regular terms in $T_{0,0,0,0}$ can be obtained in an elaborate, but straightforward, manner and the final result becomes

$$T_{0,0,0,0}/k_0^3 = \frac{9T_0^4 - 10T_0^2 + 9}{8T_0^3} - \frac{1}{k_0 h_0}\left\{\frac{\left(2v_g - c_0/2\right)^2}{c_S^2 - v_g^2} + 1\right\}. \tag{39}$$

The dispersion relation now follows in a straightforward manner from the Zakharov equation by substitution of (21) into (14). The resulting evolution equation is

$$\frac{\partial\hat{a}}{\partial t} + i\omega_0\hat{a} = -iT_{0,0,0,0}|\hat{a}|^2\hat{a} \tag{40}$$

Solving this with the Ansatz $\hat{a} = \hat{a}_0 e^{-i\Omega t}$ the result is

$$\Omega = \omega_0 + T_{0,0,0,0}|\hat{a}_0|^2. \tag{41}$$

In order to be able to compare with results obtained by Whitham (1974) the energy E of a wave train is introduced. Not surprisingly it is given by

$$E = \omega_0 |\hat{a}|^2 \tag{42}$$

Writing the dispersion relation (41) as

$$\Omega = \omega_0(k_0) + \Omega_2(k_0) \frac{k_0^2 E}{c_0} \tag{43}$$

one finds for Ω_2,

$$\Omega_2 = T_{0,0,0,0}/k_0^3, \tag{44}$$

where $T_{0,0,0,0}$ is given by Eq. (39).

Whitham (1974) derived the nonlinear dispersion relation for shallow water waves using a variational approach and his result (Eq. (16.103), see also Eq.(20)) is in exact agreement with the present result displayed in Eq. (44). Combining Whitham's analysis and our work it appears that the singular terms in $T_{1,2,3,4}$ result from the wave-induced changes in mean sea surface level and the wave-induced mean flow. These changes have a stabilizing effect on the Benjamin-Feir instability as k_0h_0 decreases from the deep water limit. The critical value for stability is determined by the value of k_0h_0 for which $\Omega_2 = 0$. This value is found numerically to be $k_0h_0 = 1.363$. For $k_0h_0 > 1.363$ modulations grow, while instability is absent in the opposite case.

This threshold for instability was deduced by Whitham from the variational approach and by Benjamin (1967) by means of a Fourier mode analysis. There are important implications for the probability distribution function (pdf) of the surface elevation. For $k_0h_0 > 1.363$ nonlinearities result in focussing of wave energy, hence the kurtosis of the pdf is positive, reflecting the increased probability of extreme waves. In the opposite case nonlinearities result in defocussing, hence the kurtosis of the pdf is negative (Janssen, 2003).

Finally, note that Resio *et al.* (2001) have considered this problem before. These authors claim that the Zakharov equation does not include wave-induced currents, while they numerically find that $T_{0,0,0,0}$ is given by the first factor in Eq. (39). We can reproduce their result by numerically taking the limit in such a way that the resonance condition $\vec{k}_1 + \vec{k}_2 = \vec{k}_3 + \vec{k}_4$ is not satisfied. We argue that it is essential to satisfy the resonance condition because this condition follows from a basic property of the basis functions, namely the orthonormality property (see also Gorman, 2003).

3.3 Consequences for wave prediction in shallow water

The threshold for instability at $k_0h_0 = 1.363$ has important consequences for wave modelling in shallow waters of intermediate depth. The reason is that for these dimensionless depths there is a considerable reduction of the nonlinear transfer and hence the shape of the wave spectrum is only determined by the balance of wind input and dissipation.

In order to illustrate the stabilising effect of the wave-induced current and mean surface elevation we have plotted in Fig. (1) the narrow band transfer coefficient $R = T_{0,0,0,0}/k_0^3$ as function of dimensionless depth using Eq. (39) with and without the wave-induced current effects. Including wave-induced effects shows that indeed the transfer coefficient changes sign at $k_0h_0 = 1.363$ while the transfer only approaches very slowly the deep water value (in agreement with the fact that the mean surface elevation slowly vanishes like $1/(k_0h_0)$ and not exponentially). In Fig.(1) we have also plotted the narrow-band approximation of the nonlinear transfer using the complete expression in Eq. (13). In order to take the limit numerically we perturbed the relevant

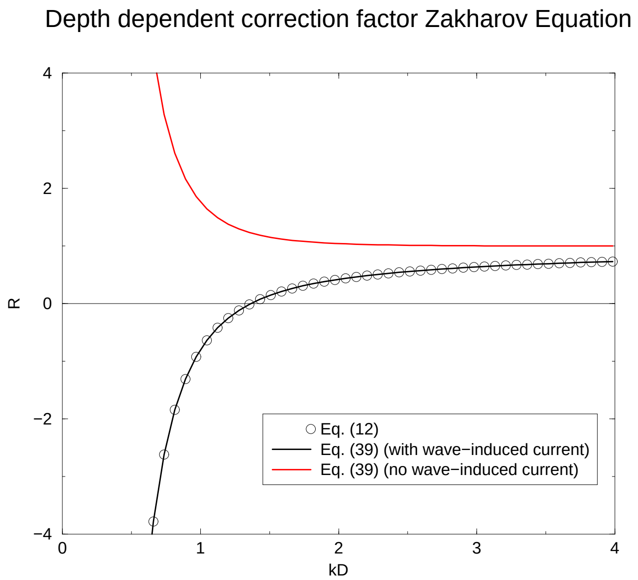

Figure 1: Depth dependence of numerical (Eq. (13)) and analytical narrow-band approximation (Eq. (39) of the nonlinear transfer coefficient normalized with the deep water value. The effect of the wave-induced current and mean surface elevation is shown as well.

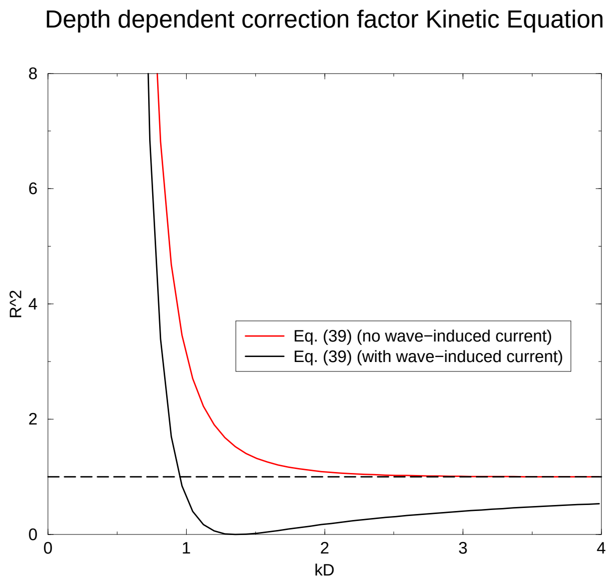

Figure 2: Depth dependence of the square of the nonlinear transfer coefficient in the narrow-band approximation (Eq. (39) in comparison with the case when wave-induced effects are removed.

wavenumbers according to Eq. (35). The agreement with the analytical result (39) is satisfactory. Hence, the Zakharov equation contains the effects of wave-induced current and mean surface elevation.

These wave-induced effects have an even more pronounced effect in the kinetic equation for the action density, as the nonlinear transfer coefficient is squared. Recall that according to Janssen (2003) the corresponding kinetic equation becomes

$$\begin{aligned}\frac{\partial}{\partial t}N_4 &= 4\int d\vec{k}_{1,2,3}T^2_{1,2,3,4}\delta(\vec{k}_1+\vec{k}_2-\vec{k}_3-\vec{k}_4)R_i(\Delta\omega,t)\\ &\quad\times\left[N_1N_2(N_3+N_4)-N_3N_4(N_1+N_2)\right],\end{aligned} \tag{45}$$

where $R_i(\Delta\omega,t)=\sin(\Delta\omega t)/\Delta\omega$ and $\Delta\omega=\omega_1+\omega_2-\omega_3-\omega_4$. In Fig. (2) we have plotted R^2 as function of dimensionless depth and compared it with the case in the absence of wave-induced effects. Clearly, in a wide range of dimensionless depth around $k_0h_0=1.363$ the nonlinear transfer is small. Although the narrow-band approximation to the nonlinear transfer only has a very restricted validity, it nevertheless indicates that the wave-induced effects should have a dramatic impact on the down shifting of the peak of the wave spectrum in shallow water. When the peak wavenumber of the spectrum approaches the threshold value $k_{thr}=1.363/h_0$ one would expect that the downshift of the spectrum is arrested.

In order to test this conjecture we simulated the evolution of the wave spectrum by performing Monte Carlo Forecasting of Eq. (14) (cf. Janssen (2003) for a number of cases namely $k_0h_0=1.363/2, k_0h_0=1.363$ and $k_0h_0=3\times1.363$. The size of the ensemble is 500, while the Benjamin Feir Index equals 1. Results for the spectrum and the nonlinear transfer are displayed in Fig. (3). Clearly, the 'deep' water simulation shows the expected down shift of the spectrum, while the intermediate water depth case ($k_0h_0=1.363$) shows no change of the spectrum at all while the shallow water case shows signs of an upshifting of the peak of the spectrum. Evidently, a simple scaling of the deep-water nonlinear transfer for shallow water cases (as is common practice in wave modelling) does not seem to be a realistic option.

From the numerical simulations we have also obtained the time evolution of the kurtosis. These results are plotted in Fig. 4 and are in agreement with our expectations. For deep-water we find a positive kurtosis (in agreement with Janssen, 2003), hence there is an increased probability for extreme events. In shallow water, on the other hand, kurtosis is found to be negative, thus it is less likely than normal to find extreme waves.

These simulations have been repeated with the kinetic equation (45) and for $k_0h_0\geq1.363$ a good agreement with the Monte Carlo Simulations is found. For $k_0h_0<1.363$ we have defocussing, and Janssen (2003) found that in those circumstances the range of validity of the kinetic equation is much restricted: by performing Monte Carlo Simulations with the NLS equation good agreement was only found for $BFI<0.5$. However, from Fig. (1) we see that in shallow water the nonlinear transfer coefficient increases very rapidly with decreasing dimensionless depth, so we very quickly end up with a strongly nonlinear case.

Referring to Janssen (2003) where the properties of the Zakharov equation were discussed, it was argued that one is basically studying the balance between dispersion and nonlinearity. Thus, balancing the nonlinear term and the dispersive term in the narrow-band version of Eq.(14) therefore gives the dimensionless number

$$-\frac{v_g^2}{c^2}\frac{gT_0}{\omega_0}\frac{1}{k_0^4\omega_0''}\frac{s^2}{\sigma_\omega'^2}. \tag{46}$$

Since our interest is in the dynamics of a continuous spectrum of waves the slope parameter s and the relative width σ'_ω of the frequency spectrum relate to spectral properties, hence $s=(k_0^2<\eta^2>)^{\frac{1}{2}}$, with $<\eta^2>$ the average surface elevation variance, and $\sigma'_\omega=\sigma_\omega/\omega_0$. For positive sign of the dimensionless parameter (46) there is focussing (modulational instability) while in the opposite case there is defocussing of the weakly nonlinear wave train.

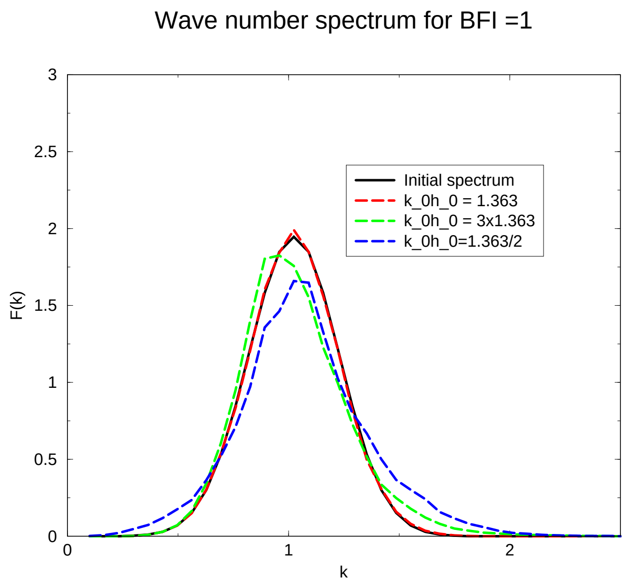

Figure 3: Spectral evolution of shallow-water case ($k_0h_0 = 1.363/2$), an intermediate depth case ($k_0h_0 = 1.363$) and a 'deep'-water case ($k_0h_0 = 3 \times 1.363$), showing upshifting and downshifting of the spectrum respectively caused by nonlinear interactions. the $BFI = 1$. The spectra have been scaled in such a manner that the total surface is 1.

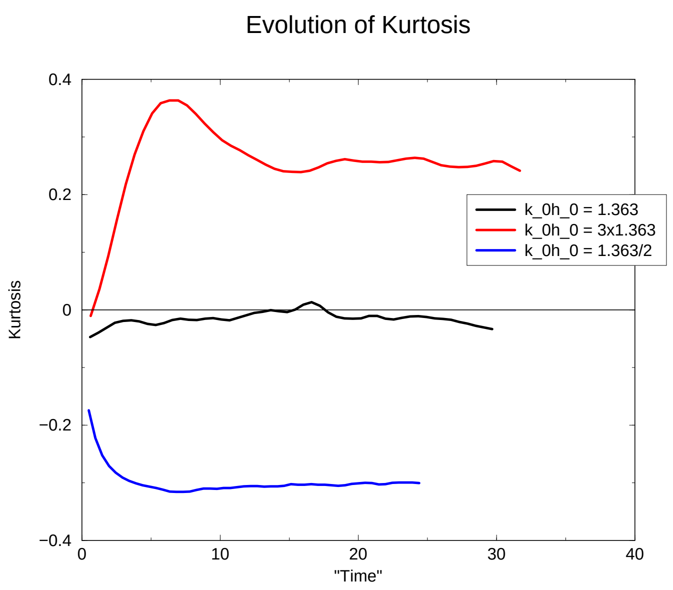

Figure 4: Time evolution of kurtosis for $BFI = 1$. As expected for deep-water($k_0h_0 = 3 \times 1.363$) nonlinearity focusses waves resulting in a positive kurtosis while for shallow water($k_0h_0 < 1.363$) we have defocussing giving a negative kurtosis.

Using the dispersion relation for deep-water gravity waves and the expression for the nonlinear interaction coefficient, $T_0 = k_0^3$, the BF Index (which is the square root of (46)) becomes,

$$BFI = \frac{s\sqrt{2}}{\sigma'_\omega}. \tag{47}$$

However, in the general shallow water case the appropriate dimensionless parameter becomes

$$BFI = \frac{s\sqrt{2}}{\sigma'_\omega} \frac{2v_g}{c} \sqrt{|T_0|/k_0^3}. \tag{48}$$

Hence, for $k_0 h_0 < 1.363$ the factor T_0 increases rapidly with decreasing depth and one quickly deals with a strongly nonlinear problem. The kinetic equation is then only valid for rapidly decreasing ratio of steepness and relative width.

4 Conclusions

The overall conclusions from this work are clear. The threshold value for instability $k_0 h_0 = 1.363$ plays an important role in understanding the generation of freak waves and in understanding the spectral evolution in shallow water. A simple scaling of the deep-water nonlinear transfer for shallow water cases is probably not working.

The main concern is now why all this has not been noticed before. Nonlinear energy transfer in intermediate water depths has been studied before. Herterich and Hasselmann (1980) also mention that the shallow-water energy transfer cannot be scaled using the deep-water transfer, but according to these authors this occurs for much smaller values of $k_0 h_0$ namely around the value 0.7 and not around the value 1.363 as found in this work. For these very small values of $k_0 h_0$ the perturbation approach breaks down and Herterich and Hasselmann (1980) do not discuss this problem any further. However, there is a large body of literature from the 1960's pointing out that the narrow-band approximation to the nonlinear transfer vanishes at $k_0 h_0 = 1.363$. For these values of dimensionless depth the perturbation approach is appropriate, and therefore one should deal with the non-scaling behaviour of the shallow water nonlinear energy transfer. In addition, wind waves at these dimensionless depths are a common feature near oil riggs and buoys, hence use of a more appropriate scaling factor, e.g. the one from Eq. (39), should be investigated.

A Evaluation of the potential for a single wave.

Let $\hat{\phi}$ be the Fourier transform of the velocity potential ϕ and let $\hat{\psi}$ be the Fourier transform of the value of the potential at the surface.. To second order in amplitude one then finds the following relation between $\hat{\phi}$ and $\hat{\psi}$:

$$\hat{\phi}_1 = \tanh(k_1 h_0)\left[\hat{\psi}_1 - \int d\vec{k}_{2,3} q_2 \hat{\psi}_2 \hat{\eta}_3 \delta_{1-2-3}\right]. \tag{A1}$$

The velocity potential is then given by

$$\phi(x) = \int d\vec{k}\, \hat{\phi}(\vec{k}) \frac{\cosh(k(z+h_0))}{\sinh(kh_0)}\, e^{i\vec{k}.\vec{x}} \tag{A2}$$

and using (A1) the potential at z=0, the mean surface, becomes

$$\phi(x) = \int d\vec{k}\, \hat{\psi}(\vec{k}) e^{i\vec{k}.\vec{x}} - \int d\vec{k}_{1,2,3} e^{i\vec{k}_1.\vec{x}} q_2 \hat{\psi}_2 \hat{\eta}_3 \delta_{1-2-3} \tag{A3}$$

Note that $\hat{\psi}$ and $\hat{\eta}$ are given in terms of the action variable $A(\vec{k})$ by Eq. (4), while for a single wave the action variable is given by Eq. (29). There are two contributions to $\phi(x)$, which are denoted by $\mathscr{A}$ and $\mathscr{B}$.

The first one, $\mathscr{A}$, is given by

$$\mathscr{A} = \int d\vec{k}\, \hat{\psi}(\vec{k}) e^{i\vec{k}.\vec{x}} = -i \int d\vec{k} \sqrt{\frac{g}{2\omega}}\, A(\vec{k}) e^{i\vec{k}.\vec{x}} + \text{c.c} \tag{A4}$$

Making use of Eq. (29) and the introduction of the amplitude a (cf. below (29)) one finds in the limit of small $\vec{\varepsilon}$

$$\begin{aligned}\mathscr{A} = {} & \frac{ga}{\omega_0} \sin\theta - \frac{ga^2}{\omega_0}\sqrt{\frac{g}{2\omega_2}}\left[B_2(2\vec{k}_0) - B_{-2}(-2\vec{k}_0)\right] \sin 2\theta \\ & - \frac{1}{2}\frac{ga^2}{\omega_0}\sqrt{\frac{g}{2\omega(\varepsilon)}}\left[B_0(\vec{\varepsilon}) - B_0(-\vec{\varepsilon})\right] \sin \vec{\varepsilon}.\vec{x}.\end{aligned} \tag{A5}$$

thus giving a linear oscillation, a second harmonic and a mean flow contribution.

The second one, $\mathscr{B}$, reads

$$\mathscr{B} = -\int d\vec{k}_{1,2,3} e^{i\vec{k}_1.\vec{x}} q_2 \hat{\psi}_2 \hat{\eta}_3 \delta_{1-2-3} \tag{A6}$$

This is already quadratic in amplitude so only the linear representation of $\hat{\psi}$ and $\hat{\eta}$ is required. As a result one finds

$$\mathscr{B} = -\frac{1}{2}\frac{ga^2}{\omega_0} q_0 \sin 2\theta. \tag{A7}$$

Combining the two one finds for the potential of a single wave

$$\begin{aligned}\phi(x) = {} & \frac{ga}{\omega_0}\sin\theta - \frac{ga^2}{\omega_0}\sin 2\theta\left[\sqrt{\frac{g}{2\omega_2}}\left(B_2(2\vec{k}_0) - B_{-2}(-2\vec{k}_0)\right) + \frac{1}{2}q_0\right] \\ & - \frac{1}{2}\frac{ga^2}{\omega_0}\sqrt{\frac{g}{2\omega(\varepsilon)}}\left[B_0(\vec{\varepsilon}) - B_0(-\vec{\varepsilon})\right] \sin\vec{\varepsilon}.\vec{x}.\end{aligned} \tag{A8}$$

Making use of the expressions for B_0, B_2 and B_{-2} one finds in the limit of small $\vec{\varepsilon}$

$$\phi(x) = \beta x + \frac{ga}{\omega_0}\sin\theta + \nu a^2 \sin 2\theta \tag{A9}$$

where

$$\nu = \frac{3}{8}\frac{\omega_0}{T_0^4}\left(1 - T_0^4\right), \tag{A10}$$

while

$$\beta = -\frac{1}{4}\frac{k_0 a^2}{c_S^2 - v_g^2}\left[\frac{g v_g}{T_0}\left(1 - T_0^2\right) + \frac{2g^2}{\omega_0}\right]. \tag{A11}$$

Note that Eq. (A10) is in complete accord with Whitham (1974, 13.123 for $z = 0$). From the variational approach Whitham finds that the mass flux involves the normal contribution of the current β and a contribution by the waves. This therefore suggests to introduce the mass transport velocity U as

$$U = \beta + u_w \tag{A12}$$

where, with $E = \frac{1}{2}ga^2$, as expected

$$u_w = \frac{E}{c_0 h_0} = \frac{1}{2}\frac{g k_0}{\omega_0 h_0}a^2 \tag{A13}$$

Whitham deduces the following relation between U and the mean elevation $b = k_0 a^2 \Delta$ (cf Eq. (33)):

$$U = \frac{v_g}{h_0}\, b \tag{A14}$$

Using (A11) in (A12) it can be verified that (A14) is indeed satisfied.

It is concluded that the narrow-band version of the Zakharov equation is in complete accord with the results of Whitham's variational approach.

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