A regime-dependent balanced control variable based on potential vorticity

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Flow-dependence in data assimilation

- A-priori (background) information in the form of a forecast, $\mathbf{x}^b$**.**
- Flow dependent forecast error covariance matrix ($\mathbf{P_f}$ or **B**).
  - Kalman filter / EnKF ($\mathbf{P_f}$).
  - $\mathbf{MBM}^T$ in 4d-VAR.
  - Cycling of error variances.
  - Distorted grids (e.g. geostrophic co-ordinate transform).
  - Errors of the day.
  - Reduced rank Kalman filter.
  - Flow-dependent balance relationships (e.g. non-linear balance equation, omega equation).
  - Regime-dependent balance (e.g. 'PV control variable').

VAR (**B**)

A PV-based control variable

1. Brief review of control variables, χ, and control variable transforms, **K**.
2. Shortcomings of the current choice of control variables.
3. New control variables based on potential vorticity.
4. New control variable transforms for VAR, **K**.
5. Determining error statistics for the new variables, $\mathbf{K}^{-1}$.
6. Diagnostics to illustrate performance in MetO VAR.

1. Control variable transforms in VAR

VAR does not minimize a cost function in model space (1)

unfeasible

$$J(\delta\mathbf{x},\mathbf{x}^b) \;=\; \tfrac{1}{2}\delta\mathbf{x}^{\mathrm{T}}\mathbf{B}^{-1}\delta\mathbf{x} \;+\; \tfrac{1}{2}\sum_t(\mathbf{y}_t-\mathbf{h}_t(\mathbf{x}^b+\delta\mathbf{x}))^{\mathrm{T}}\mathbf{R}_t^{-1}(\mathbf{y}_t-\mathbf{h}_t(\mathbf{x}^b+\delta\mathbf{x}))$$

VAR minimizes a cost function in 'control variable' space (2)

feasible

$$J(\chi,\mathbf{x}^b) \;=\; \tfrac{1}{2}\chi^{\mathrm{T}}\chi \;+\; \tfrac{1}{2}\sum_t(\mathbf{y}_t-\mathbf{h}_t(\mathbf{x}^b+\mathbf{B}_0^{1/2}\chi))^{\mathrm{T}}\mathbf{R}_t^{-1}(\mathbf{y}_t-\mathbf{h}_t(\mathbf{x}^b+\mathbf{B}_0^{1/2}\chi))$$

$$\delta\mathbf{x}=\mathbf{B}_0^{1/2}\chi$$

model variable

control variable

control variable transform

(1) and (2) are equivalent if

$$\mathbf{B}=\mathbf{B}_0^{1/2}\mathbf{B}_0^{\mathrm{T}/2}$$

(ie implied covariances)

CVT | Inverse CVT

$$\delta\mathbf{x}=\mathbf{B}_0^{1/2}\chi$$

$$=\mathbf{K}\mathbf{B}_u^{1/2}\chi$$

parameter transform

spatial transform

$$\delta\mathbf{x}=\mathbf{K}\tilde{\chi} \qquad\qquad \tilde{\chi}=\mathbf{K}^{-1}\delta\mathbf{x}$$

1. Control variable transforms in VAR
Example parameter transforms

ECMWF (Derber & Bouttier 1999)

$$\underbrace{\begin{pmatrix} \delta\zeta \\ \delta\eta \\ \delta(\mathrm{T},\mathrm{p_s}) \\ \delta q \end{pmatrix}}_{\delta\mathbf{x}} = \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ \mathbf{MH} & 1 & 0 & 0 \\ \mathbf{NH} & \mathbf{P} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\mathbf{K}} \underbrace{\begin{pmatrix} \delta\tilde{\zeta} \\ \delta\tilde{\eta}_r \\ \delta(\tilde{\mathrm{T}},\tilde{\mathrm{p}}_\mathrm{s})_r \\ \delta\tilde{q} \end{pmatrix}}_{\tilde{\chi}}$$

Met Office (Lorenc et al. 2000)

$$\underbrace{\begin{pmatrix} \delta\psi \\ \delta\chi \\ \delta p \\ \delta T \\ \delta q \end{pmatrix}}_{\delta\mathbf{x}} = \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \mathbf{H} & 0 & 1 & 0 \\ \mathbf{TH} & 0 & \mathbf{T} & 0 \\ (\Gamma + \mathbf{YT})\mathbf{H} & 0 & \Gamma + \mathbf{YT} & \Lambda \end{pmatrix}}_{\mathbf{K}} \underbrace{\begin{pmatrix} \delta\tilde{\psi} \\ \delta\tilde{\chi} \\ \delta\tilde{p}_r \\ \delta\tilde{\mu} \end{pmatrix}}_{\tilde{\chi}}$$

'parameter transform', $\mathbf{U_p}$

- The leading control parameters ($\delta\tilde{\zeta}$ or $\delta\tilde{\psi}$) are referred to as 'balanced' (proxy for PV).
- Balance relations are built into the problem.

The fundamental assumption is that $\delta\tilde{\zeta}$ and $\delta\tilde{\psi}$ have no unbalanced components (there is no such thing as unbalanced rotational wind in these schemes).
The **balanced vorticity approximation (BVA)**.

2. Shortcomings of the BVA (current control variables)

Unbalanced rot. wind is expected to be significant under some flow regimes

Introduce unbalanced components $\delta\psi = \delta\psi_b + \delta\psi_u \qquad \delta p = \delta p_b + \delta p_u$

3rd line of MetO scheme
$$\begin{cases} \delta p = \mathbf{H}\delta\psi + \delta p_r \\ \quad = \mathbf{H}\delta\psi_b + \boxed{\mathbf{H}\delta\psi_u} + \delta p_r \end{cases}$$

anomalous

Instead require $\delta p = \mathbf{H}\delta\psi_b + \delta p_u$

For illustration, introduce shallow water system

Introduce variables $\delta\psi = \delta\psi_b + \delta\psi_u \qquad \delta h = \delta h_b + \delta h_u$

Linearised shallow water potential vorticity (PV)
$$\begin{cases} \delta Q = gh\nabla^2\delta\psi - fg\delta h \\ \quad = gh\nabla^2\delta\psi_b - fg\delta h_b \qquad (1) \end{cases}$$

Linearised balance equation
$$0 = g\delta h_b - f\delta\psi_b \qquad (2)$$

2. Shortcomings of the BVA (current control variables) (cont.)

$$\delta Q = gh\nabla^2\delta\psi_b - f^2\delta\psi_b$$

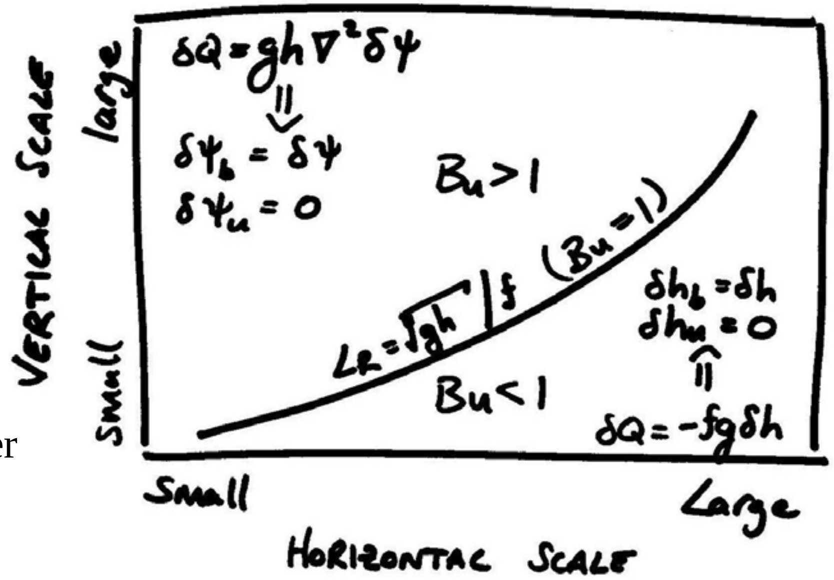
$$= f^2\left(\frac{gh}{f^2L_0^2}\hat{\nabla}^2 - 1\right)\delta\psi_b \qquad \nabla^2 = \frac{1}{L_0^2}\hat{\nabla}^2$$

$$= f^2\left(\mathrm{Bu}^2\,\hat{\nabla}^2 - 1\right)\delta\psi_b \qquad \mathrm{Bu} = \frac{L_R}{L_0}$$

wind mass

$$= \frac{\sqrt{gh}}{fL_0} \quad \text{shallow water}$$

$$= \frac{Nh}{fL_0} \quad \text{3-D}$$



Regime →	**Large Bu (small horiz/large vert scales)**	**Intermediate**	**Small Bu (large horiz/small vert scales)**
Balanced variable →	Rotational wind (BVA scheme valid)	PV or equivalent variable	Mass (BVA not valid)

3. New control variables based on PV for 3-D system

For the balanced variable

$$\delta Q = \alpha_0 \nabla^2 \delta\psi_b + \beta_0 \delta p_b + \gamma_0 \frac{\partial \delta p_b}{\partial z} + \varepsilon_0 \frac{\partial^2 \delta p_b}{\partial z^2} \qquad \text{PV}$$

$$0 = \nabla \cdot (f \rho_0 \nabla \delta\psi_b) - \nabla^2 \delta p_b \qquad \text{LBE } (\mathbf{H})$$

For the unbalanced variable 1

$$\delta\chi$$

For the unbalanced variable 2

$$\delta\overline{Q} = \nabla \cdot (f \rho_0 \nabla \delta\psi_u) - \nabla^2 \delta p_u \qquad \text{anti-PV}$$

$$0 = \alpha_0 \nabla^2 \delta\psi_u + \beta_0 \delta p_u + \gamma_0 \frac{\partial \delta p_u}{\partial z} + \varepsilon_0 \frac{\partial^2 \delta p_u}{\partial z^2} \qquad \text{anti-BE } (\overline{\mathbf{H}})$$

Standard variables: $\delta\psi,\ \delta\chi,\ \delta p_r$

PV-based variables : $\delta\psi_b,\ \delta\chi,\ \delta p_u$

} variables are equivalent at large Bu

Describes the PV ($\delta\psi_b$)

Describes the divergence ($\delta\chi$)

Describes the anti-PV (δp_u)

4. New control variable transforms

Current scheme

$$\delta\mathbf{x} = \mathbf{K}\,\tilde{\chi}$$

$$\begin{pmatrix}\delta\psi\\ \delta\chi\\ \delta p\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ \mathbf{H} & 0 & 1\end{pmatrix}\begin{pmatrix}\delta\tilde{\psi}\\ \delta\tilde{\chi}\\ \delta\tilde{p}_r\end{pmatrix}$$

($\delta\tilde{\psi}$: total streamfunction; $\delta\tilde{p}_r$: residual pressure)

$$\langle\delta\mathbf{x}\delta\mathbf{x}^T\rangle = \mathbf{K}\langle\tilde{\chi}\tilde{\chi}^T\rangle\mathbf{K}^T = \mathbf{K}\mathbf{B}_{\tilde{\chi}}\mathbf{K}^T$$

$$= \begin{pmatrix}\mathbf{B}_{\delta\tilde{\psi}} & 0 & \mathbf{B}_{\delta\tilde{\psi}}\mathbf{H}^{\mathrm{T}}\\ \mathbf{0} & \mathbf{B}_{\delta\tilde{\chi}} & 0\\ \mathbf{H}\mathbf{B}_{\delta\tilde{\psi}} & 0 & \mathbf{H}\mathbf{B}_{\delta\tilde{\psi}}\mathbf{H}^{\mathrm{T}} + \mathbf{B}_{\delta\tilde{p}_r}\end{pmatrix}$$

PV-based scheme

$$\delta\mathbf{x} = \mathbf{K}\,\tilde{\chi}$$

$$\begin{pmatrix}\delta\psi\\ \delta\chi\\ \delta p\end{pmatrix} = \begin{pmatrix}1 & 0 & \overline{\mathbf{H}}\\ 0 & 1 & 0\\ \mathbf{H} & 0 & 1\end{pmatrix}\begin{pmatrix}\delta\tilde{\psi}_b\\ \delta\tilde{\chi}\\ \delta\tilde{p}_u\end{pmatrix}$$

($\overline{\mathbf{H}}$: new unbalanced rotational wind contribution; $\delta\tilde{\psi}_b$: balanced streamfunction; $\delta\tilde{p}_u$: unbalanced pressure)

$$\langle\delta\mathbf{x}\delta\mathbf{x}^T\rangle = \mathbf{K}\langle\tilde{\chi}\tilde{\chi}^T\rangle\mathbf{K}^T = \mathbf{K}\mathbf{B}_{\tilde{\chi}}\mathbf{K}^T$$

$$= \begin{pmatrix}\mathbf{B}_{\delta\tilde{\psi}_b} + \overline{\mathbf{H}}\mathbf{B}_{\tilde{p}_u}\overline{\mathbf{H}}^T & 0 & \mathbf{B}_{\delta\tilde{\psi}_b}\mathbf{H}^{\mathrm{T}} + \overline{\mathbf{H}}\mathbf{B}_{\tilde{p}_u}\\ \mathbf{0} & \mathbf{B}_{\delta\tilde{\chi}} & 0\\ \mathbf{H}\mathbf{B}_{\delta\tilde{\psi}_b} + \mathbf{B}_{\tilde{p}_u}\overline{\mathbf{H}}^T & 0 & \mathbf{H}\mathbf{B}_{\delta\tilde{\psi}_b}\mathbf{H}^{\mathrm{T}} + \mathbf{B}_{\delta\tilde{p}_u}\end{pmatrix}$$

- Are correlations between $\delta\psi_b$ and δp_u weaker than those between $\delta\psi$ and δp_r?
- How do spatial cov. of $\delta\psi_b$ differ from those of $\delta\psi$?
- How do spatial cov. of δp_u differ from those of δp_r?
- What do the implied correlations look like?

5.Determining the statistics of the new variables

For the balanced variable – use GCR solver

Potential vorticity

$$\varepsilon_0 \nabla^2 \delta\psi_b + \beta_0 \delta p_b + \gamma_0 \frac{\partial \delta p_b}{\partial z} + \varepsilon_0 \frac{\partial^2 \delta p_b}{\partial z^2} = \delta Q$$

Linear balance equation

$$\nabla \cdot (f \rho_0 \nabla \delta\psi_b) - \nabla^2 \delta p_b = 0$$

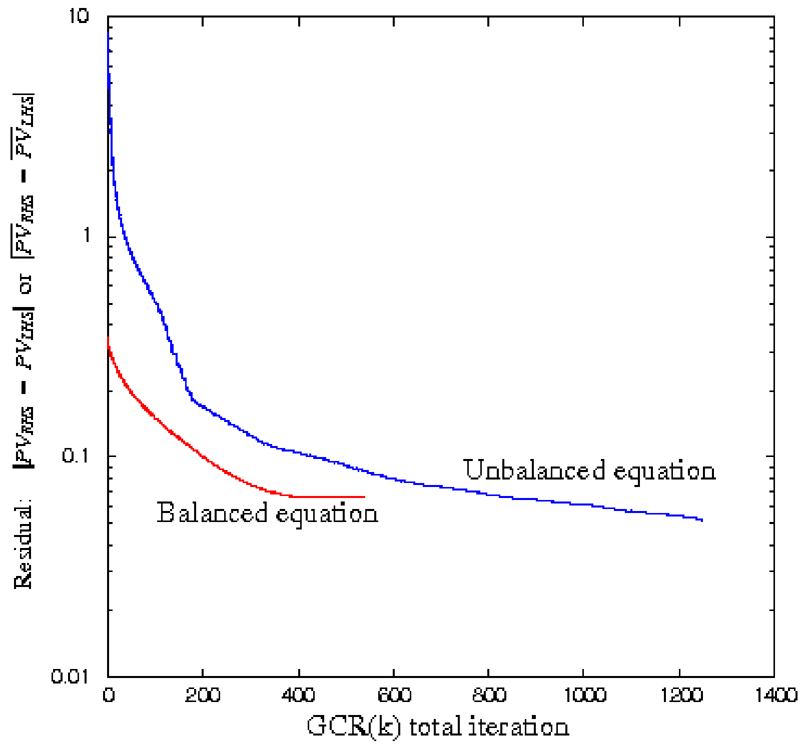
For the unbalanced variable 1 – use GCR solver

Anti-Potential vorticity

$$\nabla \cdot (f \rho_0 \nabla \delta\psi_u) - \nabla^2 \delta p_u = \delta \overline{Q}$$

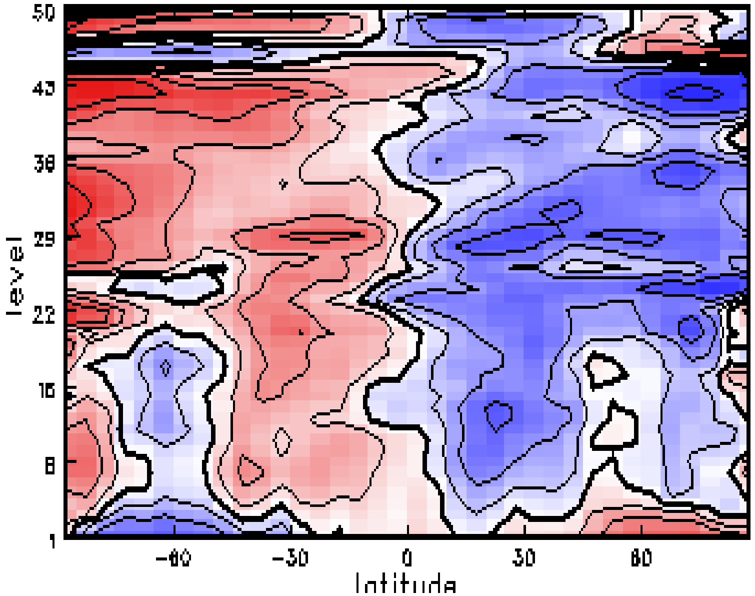
Anti-balance equation

$$\alpha_0 \nabla^2 \delta\psi_u + \beta_0 \delta p_u + \gamma_0 \frac{\partial \delta p_u}{\partial z} + \varepsilon_0 \frac{\partial^2 \delta p_u}{\partial z^2} = 0$$



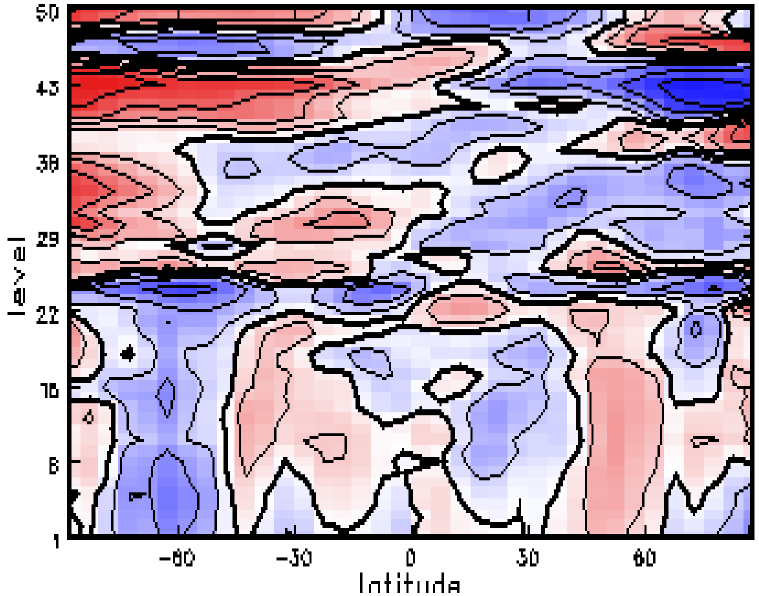
6. Diagnostics – correlations between control variables

BVA scheme: $\mathrm{cor}(\delta\psi, \delta p_r)$

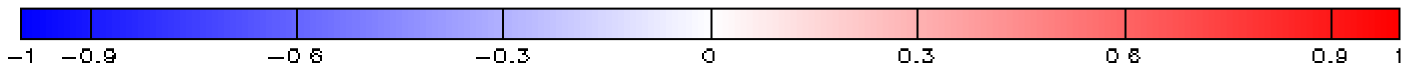


rms = 0.349

PV-based scheme: $\mathrm{cor}(\delta\psi_b, \delta p_u)$



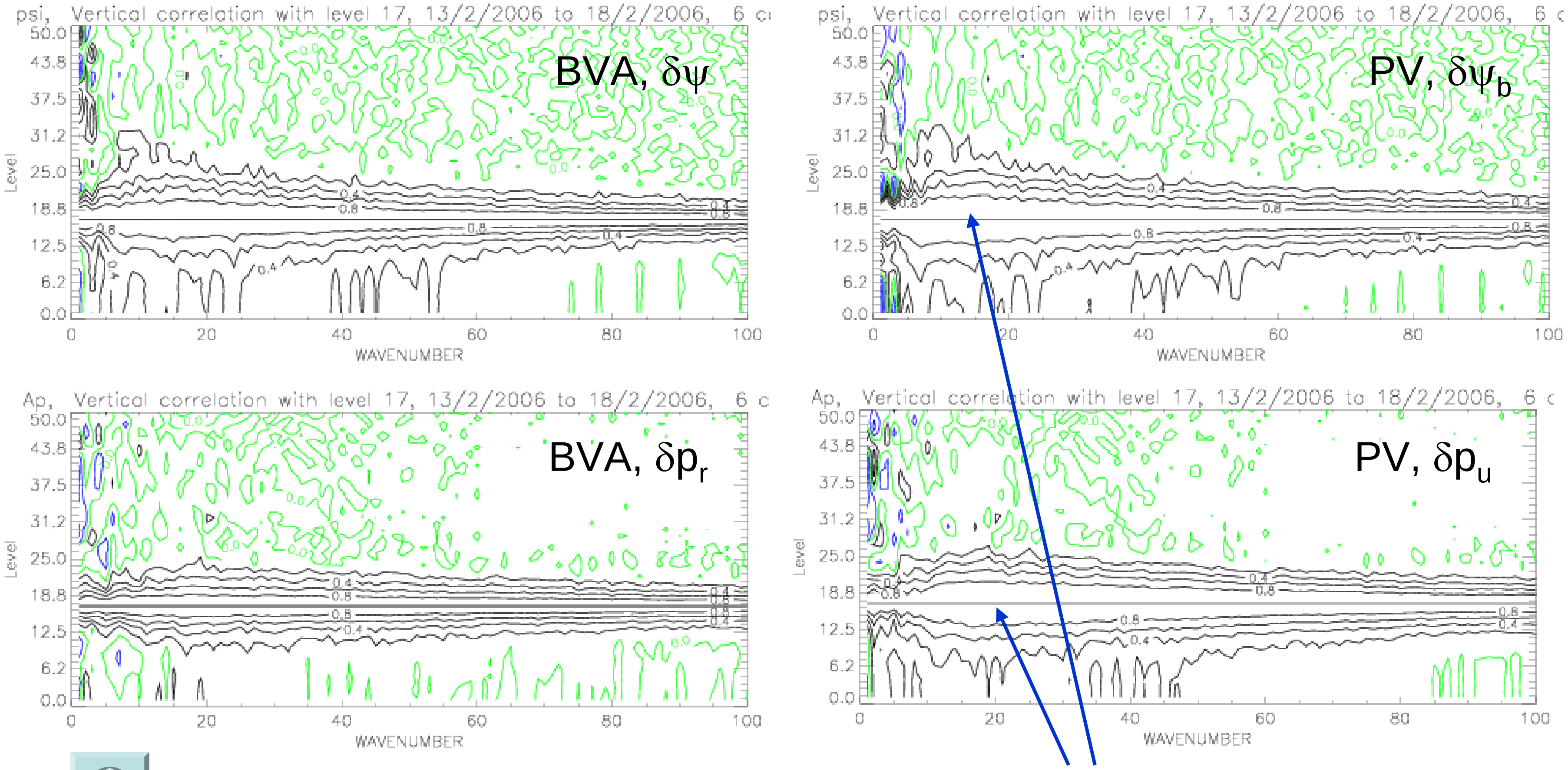
rms = 0.255



-'ve correlations, +'ve correlations

6. Diagnostics (cont) – statistics of current and PV variables (vertical correlations with 500 hPa)

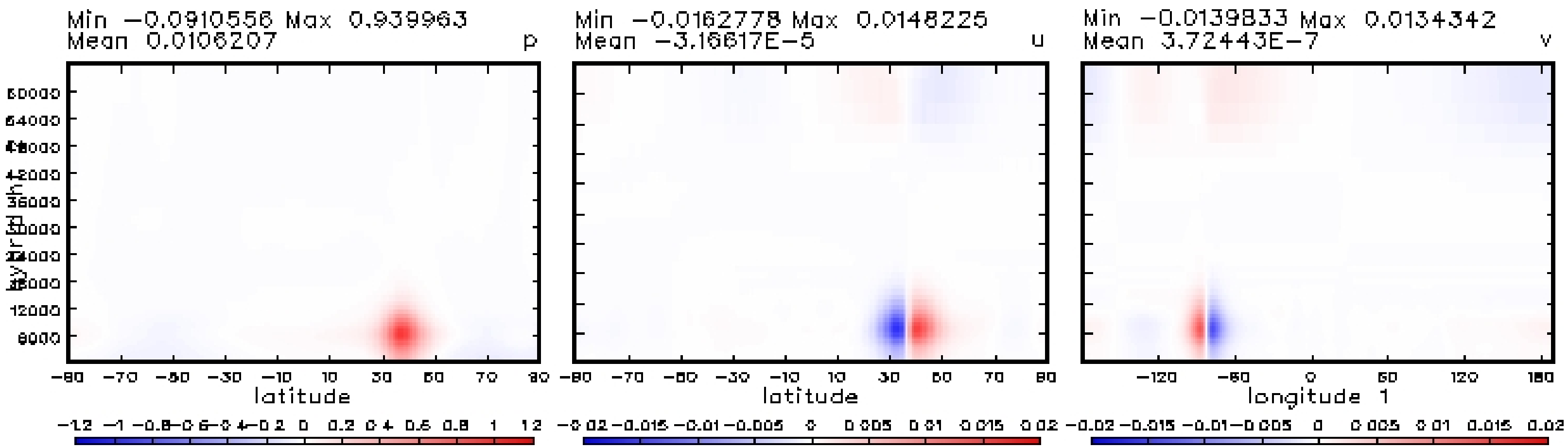
CURRENT SCHEME (BVA) **PV SCHEME**



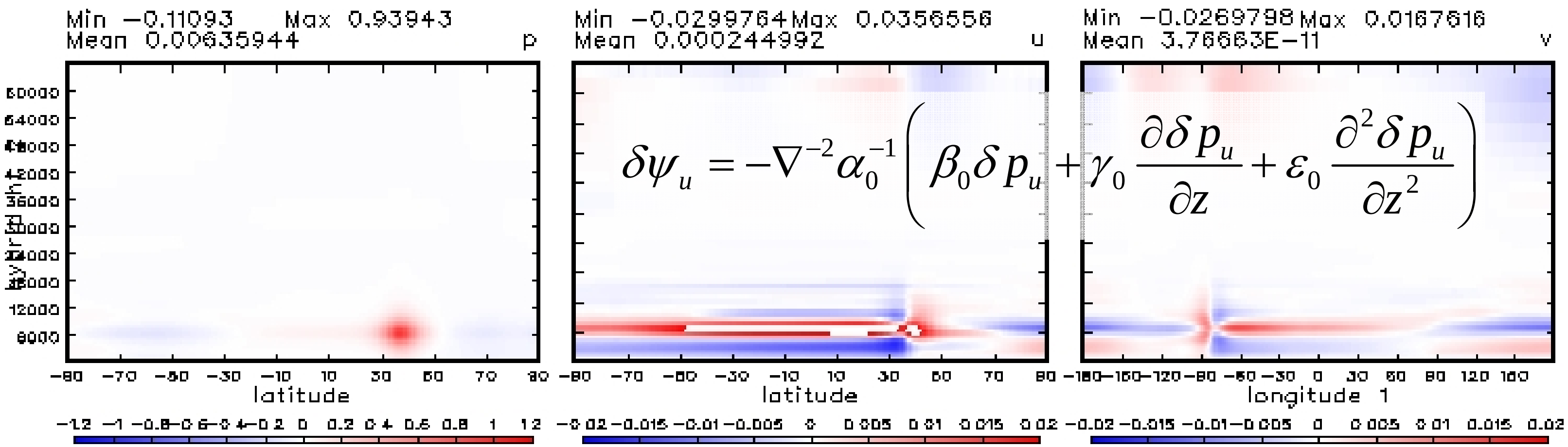
Broader vertical scales than BVA at large horizontal scales

6. Diagnostics (cont) – implied covariances from pressure pseudo observation tests

BVA scheme



PV-based scheme



$$\delta\psi_u = -\nabla^{-2}\alpha_0^{-1}\left(\beta_0\delta p_u + \gamma_0\frac{\partial\delta p_u}{\partial z} + \varepsilon_0\frac{\partial^2\delta p_u}{\partial z^2}\right)$$

Summary

- Many VAR schemes use rotational wind as the leading control variable (a proxy for PV –- the balanced vorticity approximation, BVA).
    - The BVA is invalid for small Bu regimes, $NH/fL_0 < 1$.
- Introduce new control variables.
    - PV-based balanced variable ($\delta\psi_b$).
    - anti-PV-based unbalanced variable (δp_u).
- $\delta\psi_b$ shows larger vertical scales than $\delta\psi$ at large horizontal scales.
- δp_u shows larger vertical scales than δp_r at large horizontal scales.
- $\text{cor}(\delta\psi_b, \delta p_u) < \text{cor}(\delta\psi, \delta p_r)$.
- Anti-balance equation (zero PV) amplifies features of large horiz/small vert scales in δp_u.
- The scheme is expected to work better with the Charney-Phillips than the Lorenz vertical grid.

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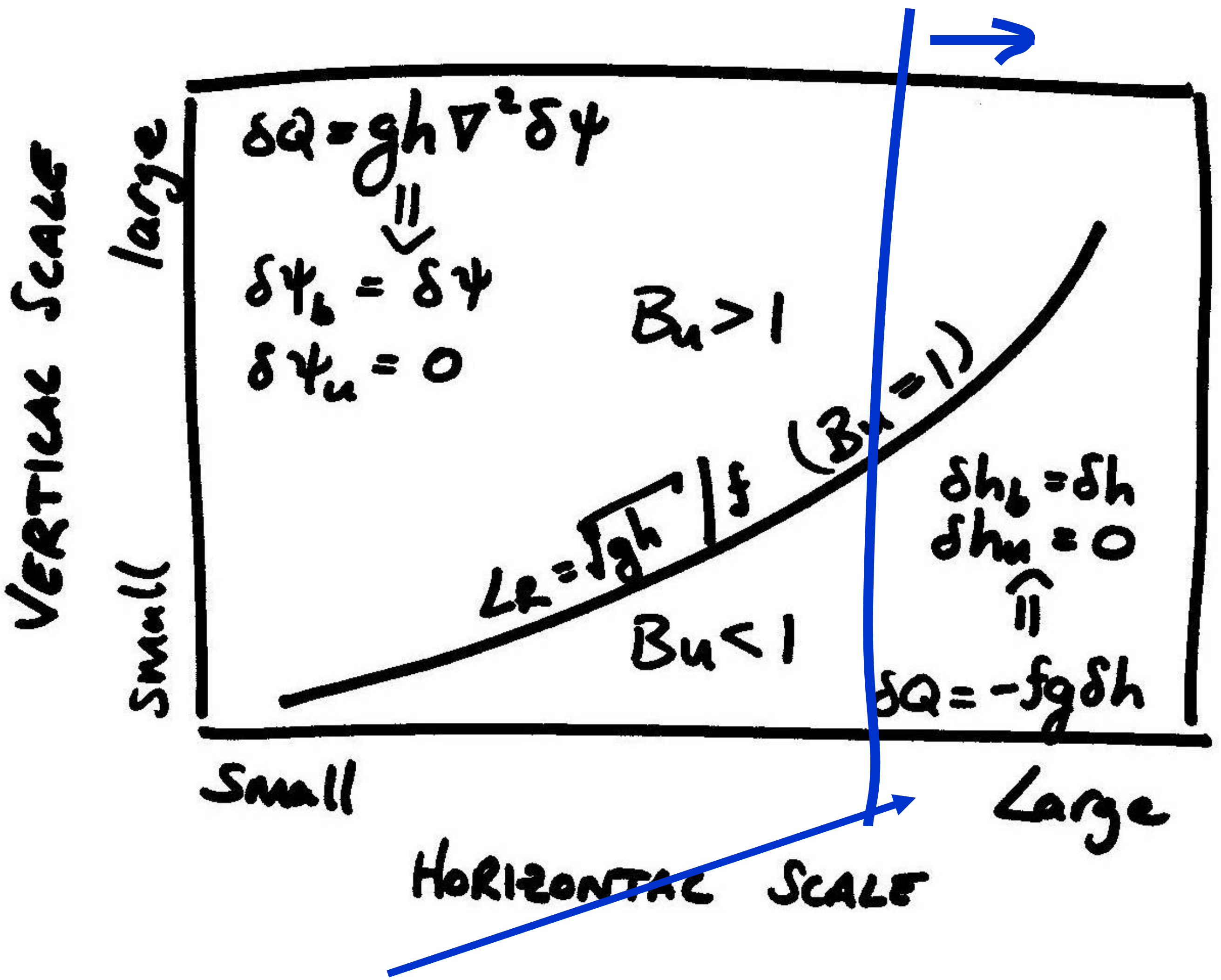
www.met.rdg.ac.uk/~ross/DARC/DataAssim.html

End



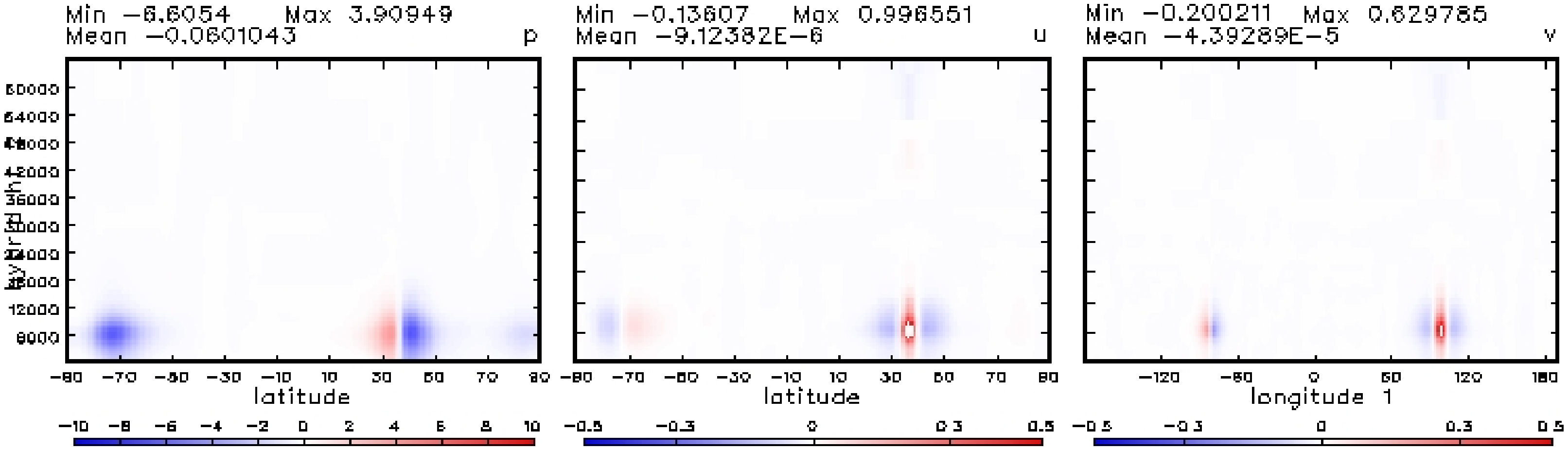
At large horizontal scales, $\delta\psi_b$ and δp_u have larger vertical scales than $\delta\psi$ and δp_r.

- Expect $\delta\psi_b < \delta\psi$
- Expect $\delta p_u \sim 0$

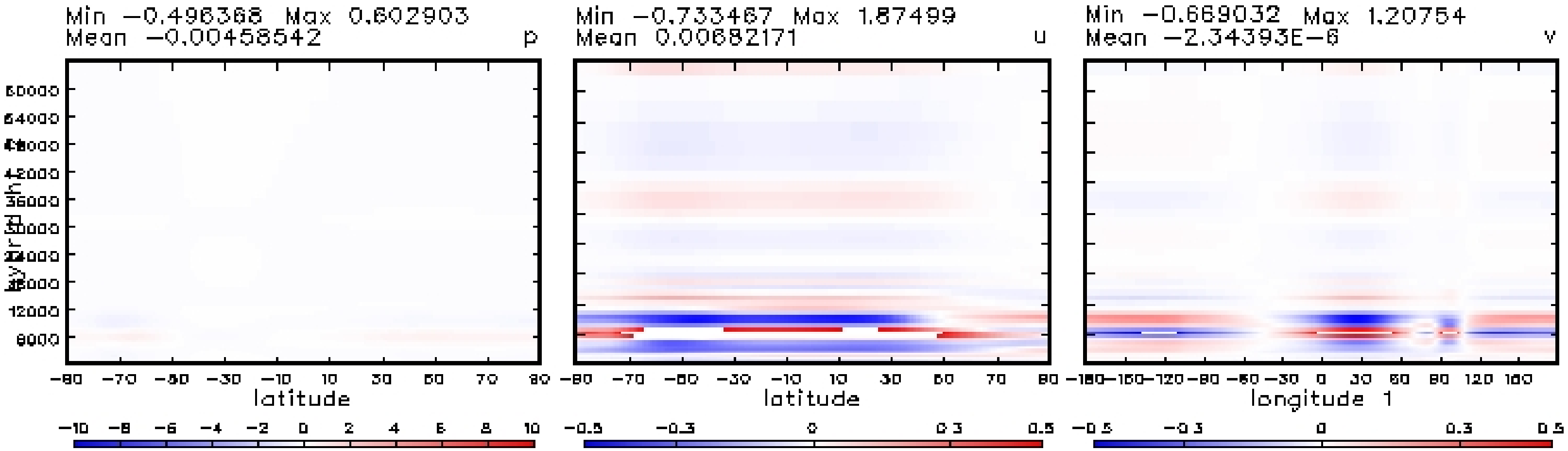
(apart from at large vertical scales).

6. Diagnostics (cont) – implied covariances from wind pseudo observation tests

BVA scheme



PV-based scheme



Actual MetO transform

$$\delta\mathbf{x} \quad = \quad \mathbf{K} \quad \tilde{\chi}$$

$$\begin{pmatrix} \delta u \\ \delta v \\ \delta p \\ \delta T \\ \delta q \end{pmatrix} = \begin{pmatrix} -\partial/\partial y & \partial/\partial x & 0 & 0 \\ \partial/\partial x & \partial/\partial y & 0 & 0 \\ \mathbf{H} & 0 & 1 & 0 \\ \mathbf{TH} & 0 & \mathbf{T} & 0 \\ (\Gamma+\mathbf{YT})\mathbf{H} & 0 & \Gamma+\mathbf{YT} & \Lambda \end{pmatrix} \begin{pmatrix} \delta\tilde{\psi} \\ \delta\tilde{\chi} \\ \delta\tilde{p}_r \\ \delta\tilde{\mu} \end{pmatrix}$$