Mixing in the SBL over Temperature-Heterogeneous Surface: LES Findings and Some Parameterization Ideas

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Outline

- LES of stably stratified PBL over temperature-homogeneous vs. temperature-heterogeneous surfaces
- Second-moment budget analysis
- Enhanced mixing in horizontally-heterogeneous SBL – an explanation
- Treatment of SBL in NWP models of DWD – problems and prospects for improvement
- Conclusions and outlook

Motivation

Models of stably stratified PBL, incl. surface layer, do not account for many important features (e.g. gravity waves, meanders of cold air, radiation flux divergence, and horizontal heterogeneity of the underlying surface)

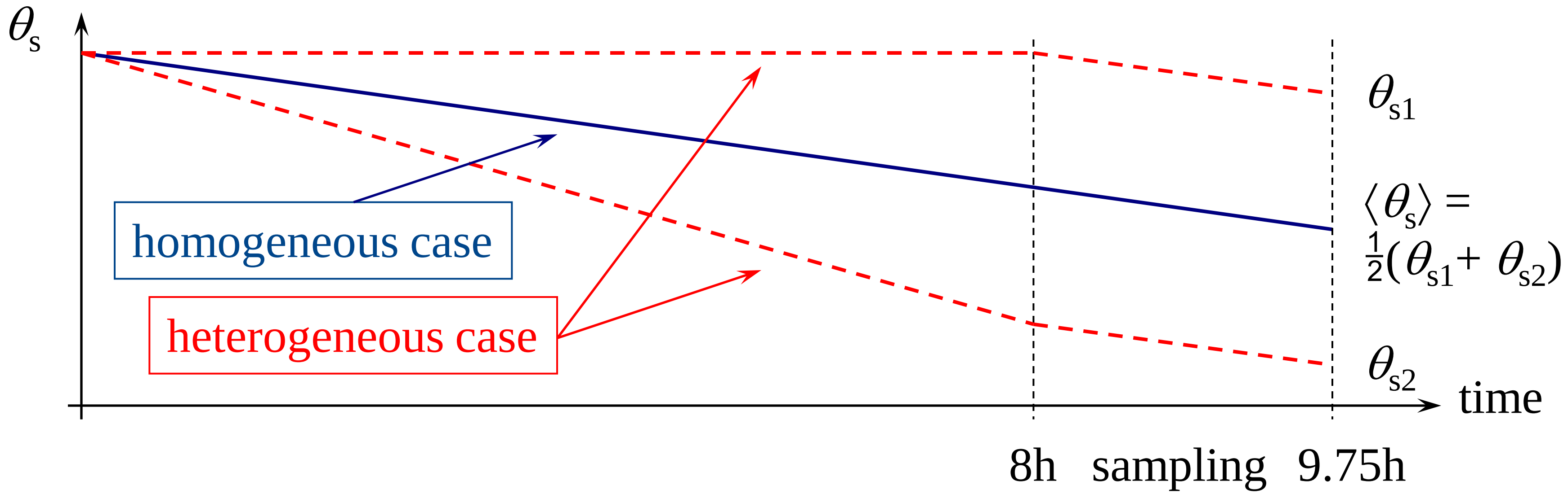
- Mixing is typically underestimated
- Models tend to quench turbulence in strongly stable stratification
- Ad hoc tuning devices like “minimum diffusion coefficients” do not help much (they are often detrimental for the NWP/climate model performance)

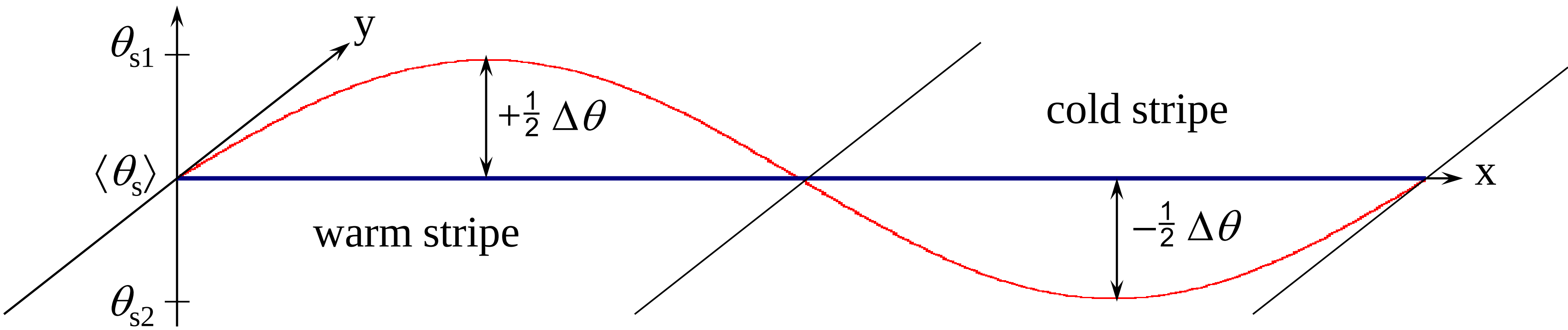
Motivation (cont'd)

- Although most turbulence models are based on truncated second-moment budget equations, no comprehensive account of second-moment budgets in stably stratified PBL (SBL), neither in horizontally-homogeneous nor in horizontally-heterogeneous case (cf. Mason and Derbyshire 1990, Coleman et al. 1992, Andrén 1995, Kosović and Curry 2000, Saiki et al. 2000, Jiménez and Cuxart 2005, Taylor and Sarkar 2008)

- Poor understanding of the role of horizontal heterogeneity in maintaining turbulent fluxes (hence no physically sound parameterisation)

Surface Temperature in Homogeneous and Heterogeneous Cases





Scalar Variance Budget Derived from LES Data (cont'd)

Adding the two budgets, we get the budget of total (resolved + SGS) scalar variance

$$\frac{1}{2}\frac{d}{dt}\left(\left\langle \bar{f}''^{2}\right\rangle + \left\langle \overline{f'^{2}}\right\rangle\right) = -\left(\left\langle \bar{u}_i''\bar{f}''\right\rangle + \left\langle \overline{u_i' f'}\right\rangle\right)\frac{\partial\left\langle \bar{f}\right\rangle}{\partial x_i} - \varepsilon_{f^2}$$

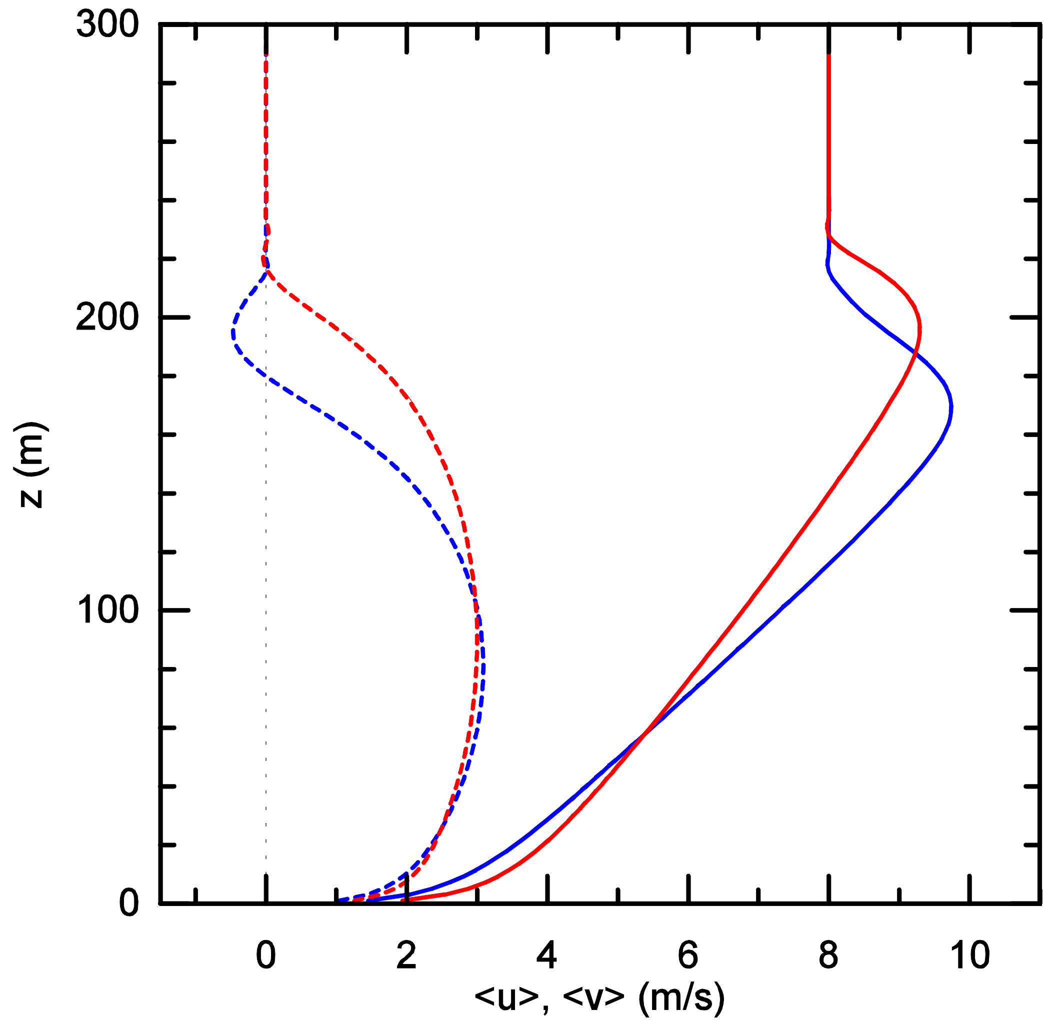
$$-\frac{1}{2}\frac{\partial}{\partial x_i}\left(\left\langle \bar{u}_i''\bar{f}''^{2}\right\rangle + \left\langle \bar{u}_i''\left(\overline{f'^{2}}\right)''\right\rangle + 2\left\langle \bar{f}''\left(\overline{u_i' f'}\right)''\right\rangle + \left\langle \overline{u_i' f'^{2}}\right\rangle\right)$$

Resolved-scale contribution

Important contributions that should be included but are neglected in most LES studies

Cannot be estimated unless high-order SGS closure model is used (presumably small)

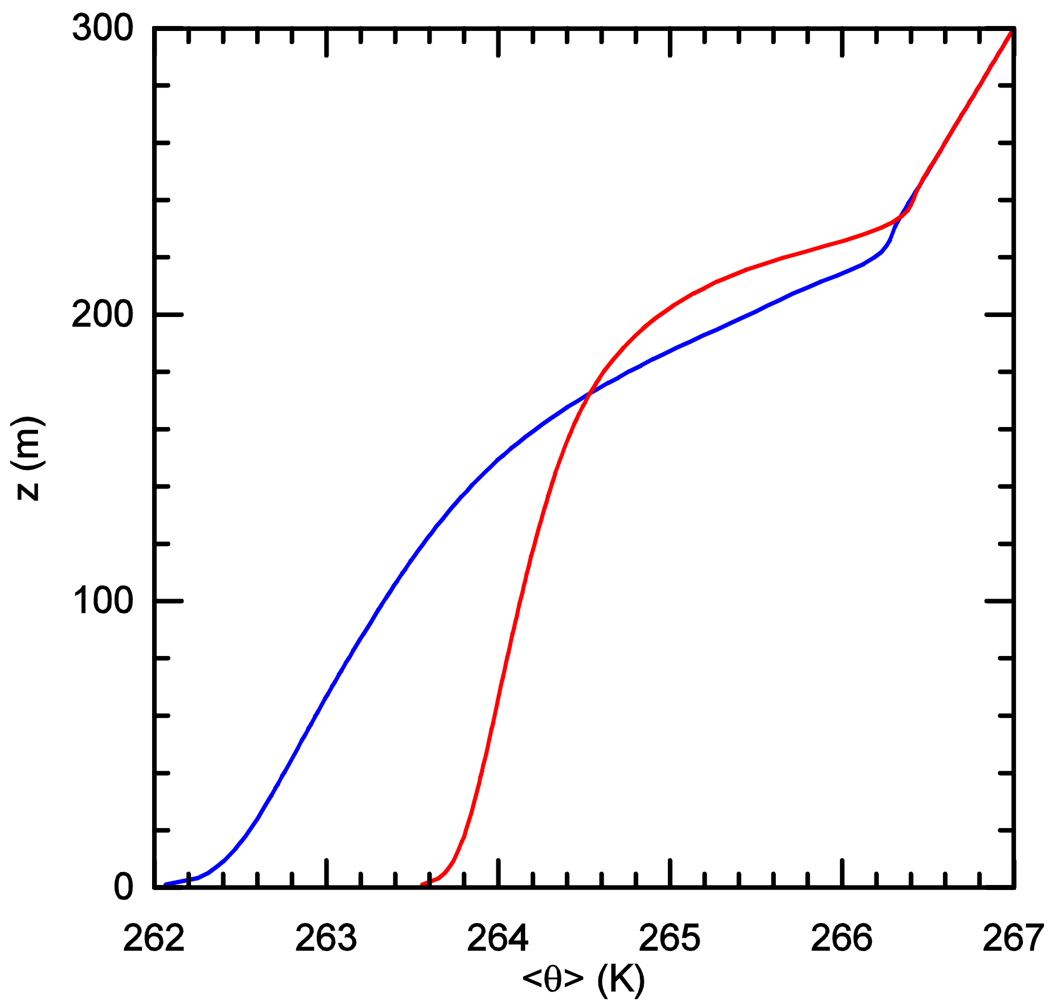
Components of Mean Wind



Blue – horizontally-homogeneous SBL,

red – horizontally-heterogeneous SBL.

Mean Potential Temperature

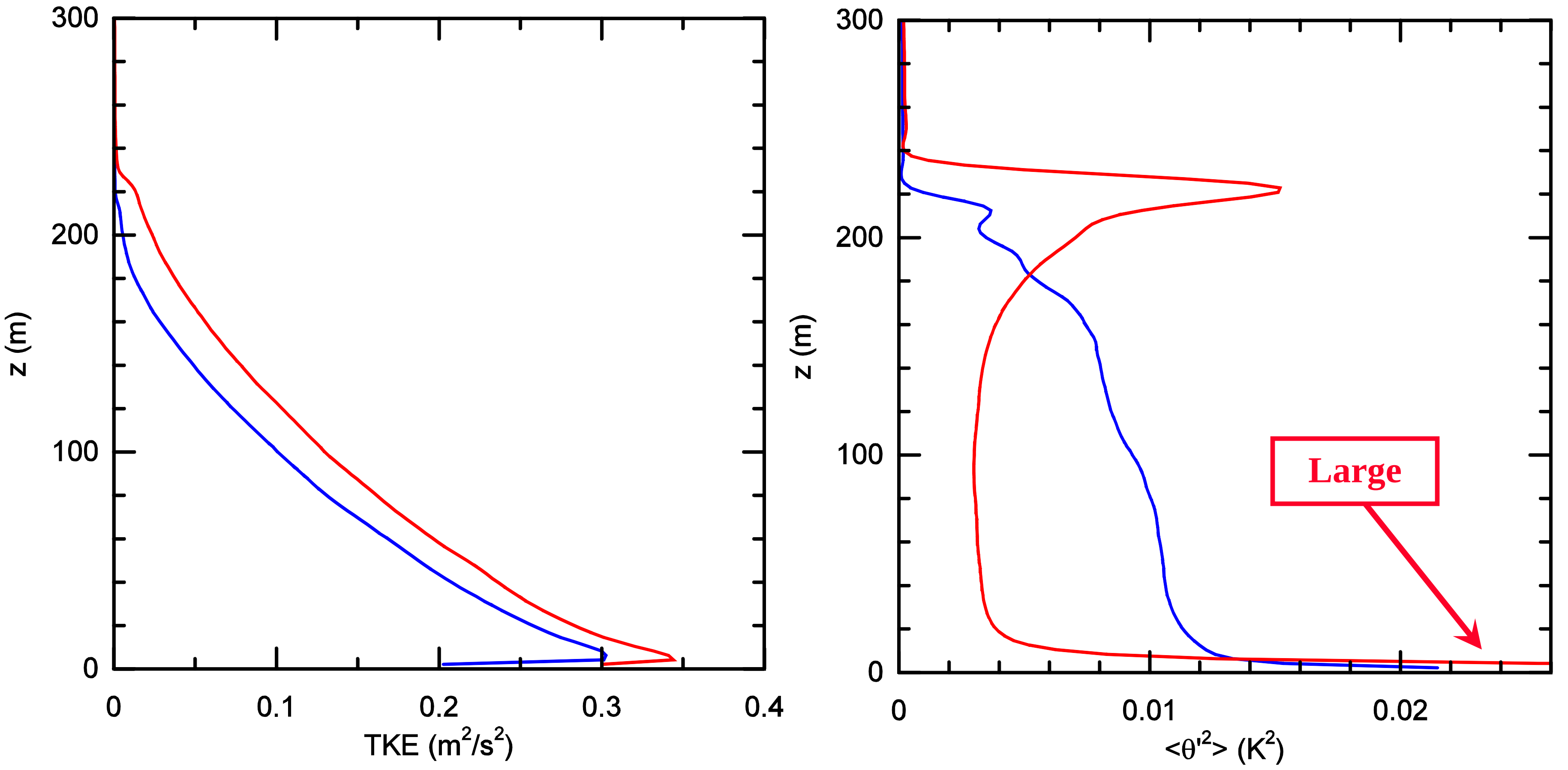


cf. Stoll and Porté-Agel (2009)

Blue – homogeneous SBL,

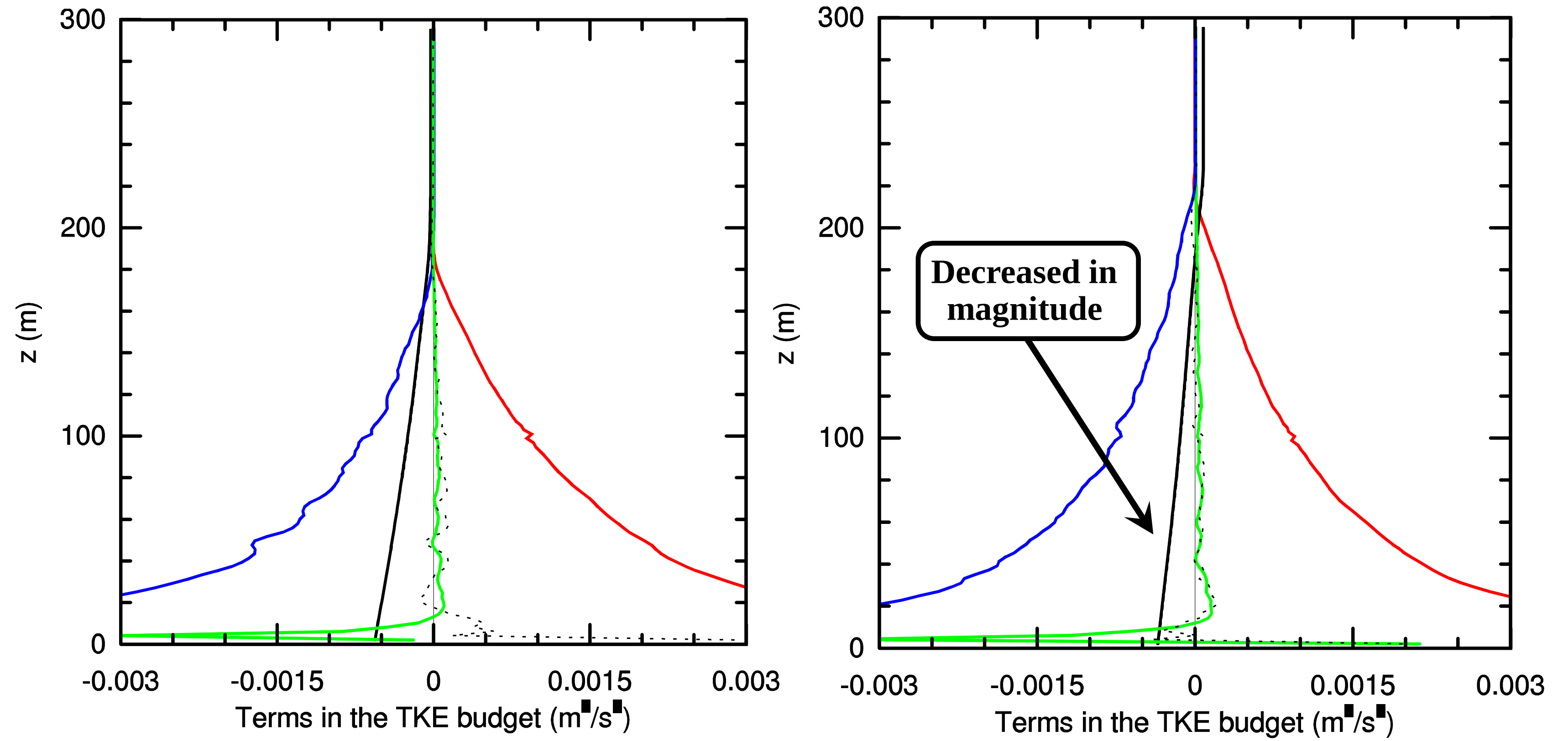
red – heterogeneous SBL.

TKE and Temperature Variance



Blue – homogeneous SBL, red – heterogeneous SBL.

TKE Budget

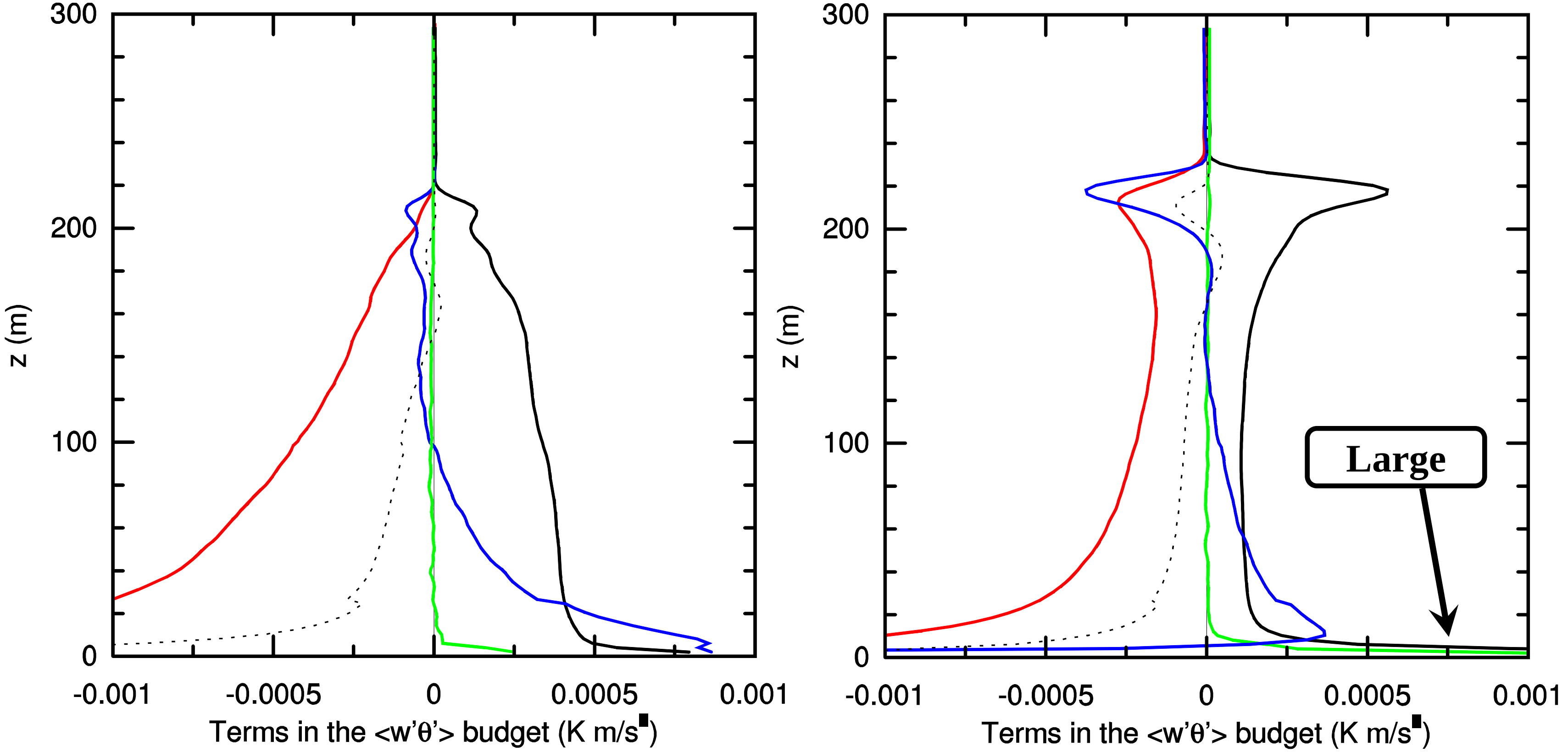


Left panel – homogeneous SBL, right panel – heterogeneous SBL.

Red – shear production, **blue** – dissipation, **black** – buoyancy destruction, **green** – third-order transport, thin dotted black – tendency .

$$\frac{\partial e}{\partial t} = -\left(\overline{w'u'}\frac{\partial \bar{u}}{\partial z} + \overline{w'v'}\frac{\partial \bar{v}}{\partial z}\right) + g\alpha\overline{w'\theta'} - \frac{\partial}{\partial z}\left(\frac{1}{2}\overline{w'u_i'^2} + \overline{w'p'}\right) - \varepsilon$$

Vertical Temperature Flux Budget

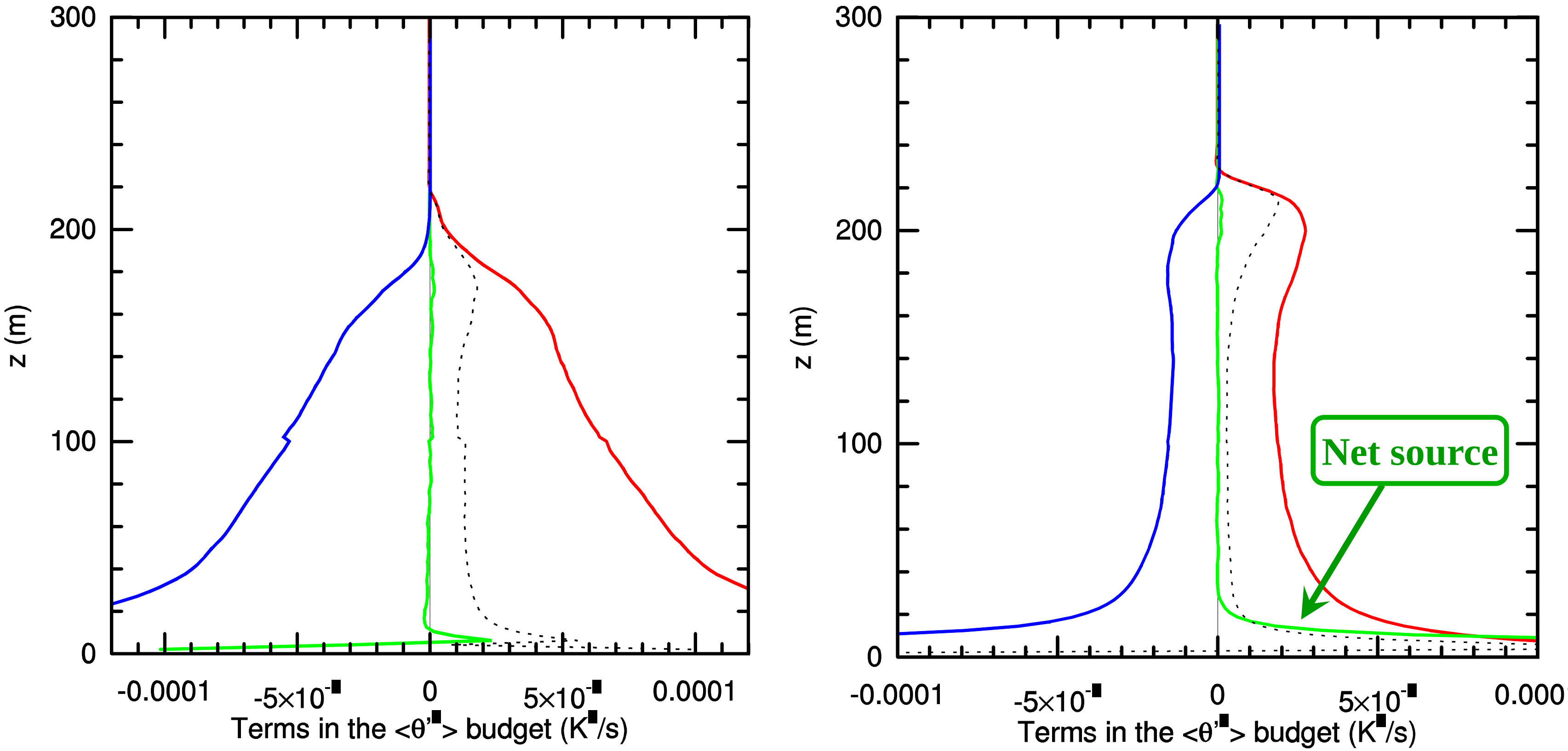


Left panel – homogeneous SBL, right panel – heterogeneous SBL.

Red – mean-gradient, **black** – buoyancy, **blue** – pressure gradient-temperature covariance, **green** – third-order transport, thin dotted black – tendency .

$$\frac{\partial \overline{w'\theta'}}{\partial t} = -\overline{w'^2}\frac{\partial \overline{\theta}}{\partial z} + g\alpha\overline{\theta'^2} - \frac{\partial}{\partial z}\overline{w'^2\theta'} - \overline{\theta'\frac{\partial p'}{\partial z}}$$

Temperature Variance Budget



Left panel – homogeneous SBL, right panel – heterogeneous SBL.

Red – mean-gradient production/destruction, blue – dissipation, green – third-order transport, black (thin dotted) – tendency .

$$\frac{1}{2}\frac{\partial \overline{\theta'^2}}{\partial t} = -\overline{w'\theta'}\frac{\partial\overline{\theta}}{\partial z} - \frac{1}{2}\frac{\partial}{\partial z}\overline{w'\theta'^2} - \varepsilon_\theta$$

Key Point: Third-Order Transport of Temperature Variance

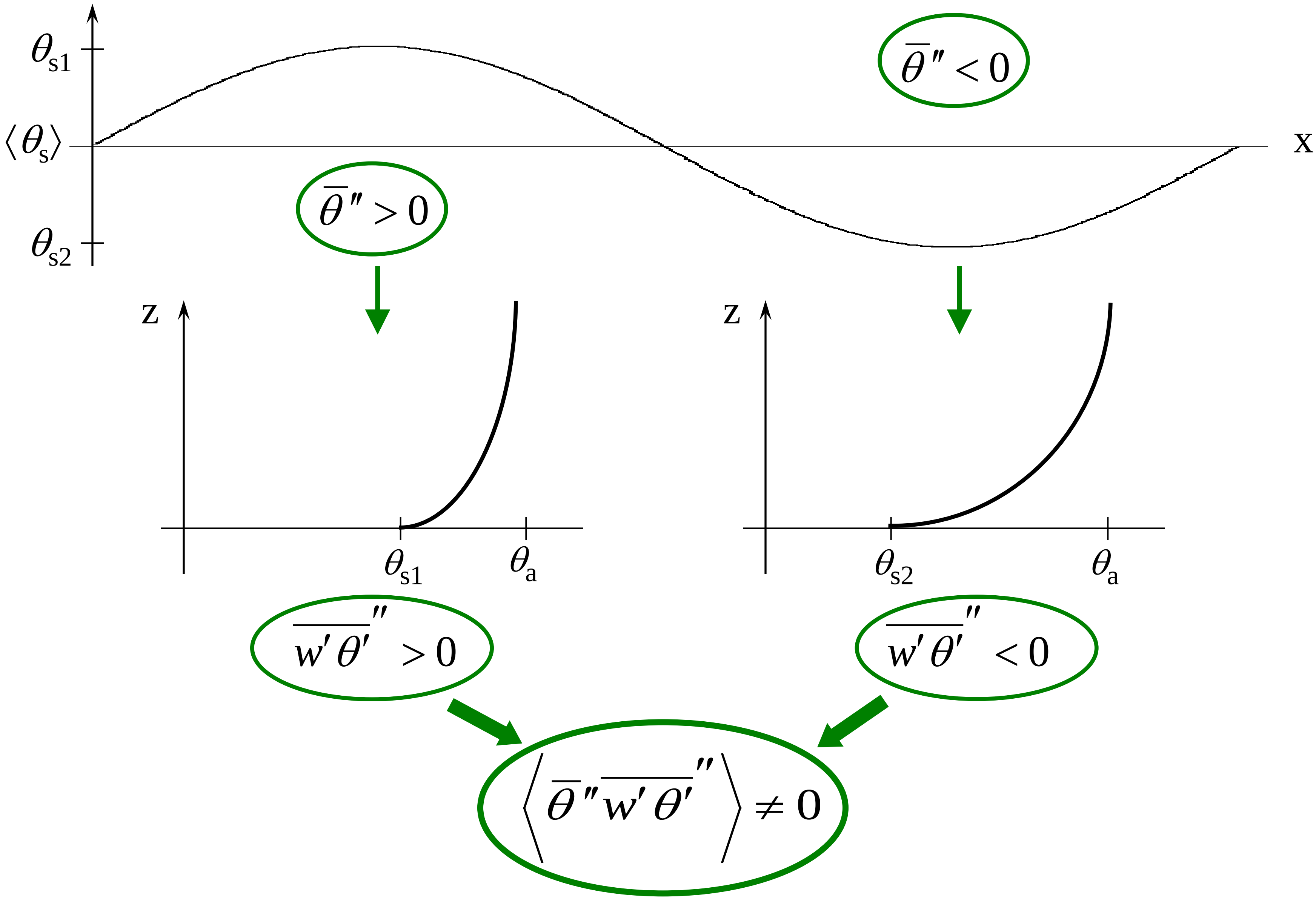
LES estimate of **<*w'θ'*²>** (resolved plus SGS)

$$\left\langle \overline{w}'' \overline{\theta}''^{2} \right\rangle + \left\langle \overline{w}'' \overline{\overline{\theta'^{2}}''} \right\rangle + 2\left\langle \overline{\theta}'' \overline{\overline{w'\theta'}''} \right\rangle + \left\langle \overline{w'\theta'^{2}} \right\rangle$$

Surface temperature variations modulate local static stability and hence the surface heat flux $\rightarrow$ net production/destruction of $<\theta'^2>$ due to divergence of third-order transport term!

In heterogeneous SBL, the third-order transport of temperature variance is non-zero at the surface

Third-Order Transport of Temperature Variance



Vertical Temperature Flux: Algebraic Closure

$$-\overline{w'^2}\frac{\partial\overline{\theta}}{\partial z}+g\alpha\overline{\theta'^2}-\cancel{\frac{\partial}{\partial z}\overline{w'^2\theta'}}-\overline{\theta'\frac{\partial p'}{\partial z}}\approx 0$$

(*) Using Rotta-type return-to-isotropy model + linear model of the so-called fast terms

$$\overline{\theta'\frac{\partial p'}{\partial z}}=\frac{\overline{w'\theta'}}{\tau}+C_b g\alpha\overline{\theta'^2}$$

(*) Neglecting anisotropy

$$\overline{u_i'u_k'}\equiv\left(\overline{u_i'u_k'}-\frac{1}{3}\delta_{ik}\overline{u_j'u_j'}\right)+\frac{1}{3}\delta_{ik}\overline{u_j'u_j'}\rightarrow\frac{2}{3}\delta_{ik}e$$

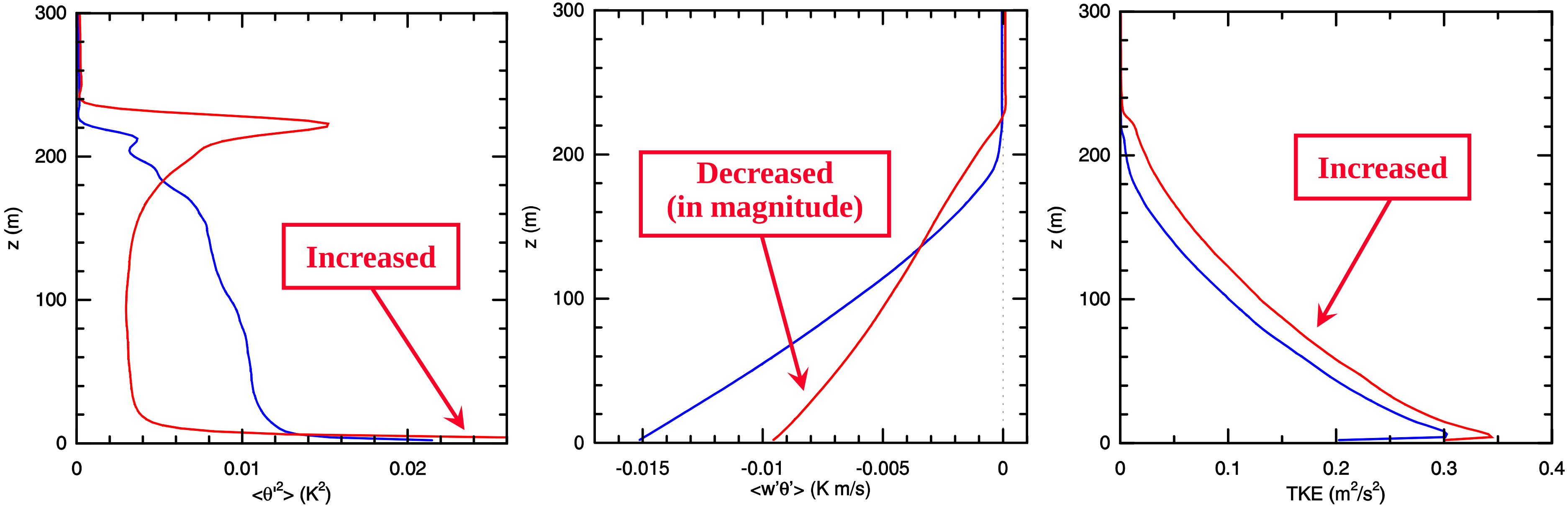
$$\overline{w'\theta'}=-\frac{2}{3}\tau e\frac{\partial\overline{\theta}}{\partial z}+(1-C_b)\tau g\alpha\overline{\theta'^2}$$

Mean-gradient production of downward (negative) temperature flux

Buoyancy production of upward (positive) temperature flux

Enhanced Mixing in Horizontally-Heterogeneous SBL
An Explanation

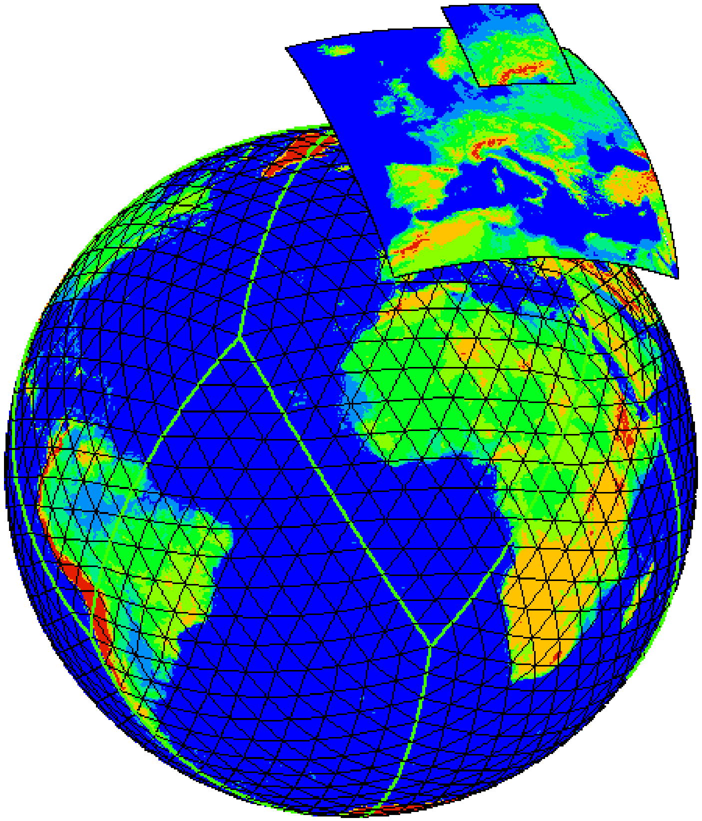
increased $\langle\theta'^2\rangle$ near the surface → reduced magnitude of downward heat flux → less work against the gravity → increased TKE → stronger mixing



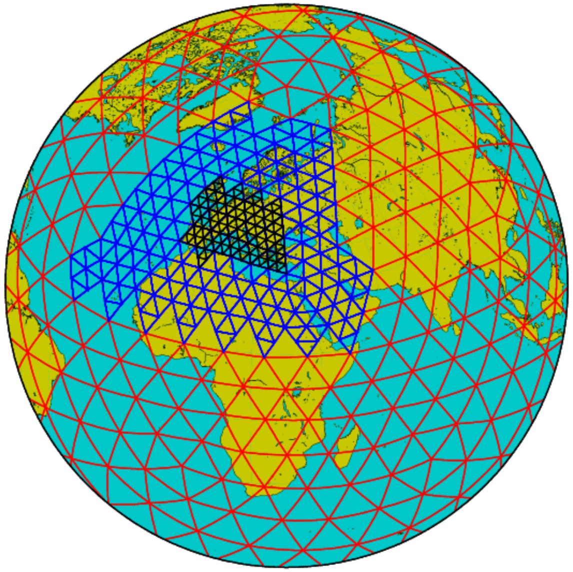
NWP Model Suite of DWD

- **GME** (Majewski et al. 2002): global, hydrostatic, icosahedral-hexagonal grid, horizontal mesh size ~30 km (~20 km), 60 vertical levels
- **COSMO-EU** (Steppeler et al. 2003, http://www.cosmo-model.org): limited-area, non-hydrostatic (fully compressible), rotated lat-lon grid, horizontal mesh size ~7 km, 40 levels, parameterisation (mass-flux) scheme of deep precipitating convection
- **COSMO-DE** (Baldauf et al. 2011, http://www.cosmo-model.org): limited-area, non-hydrostatic (fully compressible), rotated lat-lon grid, horizontal mesh size ~2.8 km, 50 levels, no parameterisation scheme of deep precipitating convection
- **ICON** (being developed by DWD and MPI-M): global, non-hydrostatic, icosahedral-triangular grid, local mesh refinement (horizontal mesh size from ~20 km to ~5 km in the focus area)

NWP Model Suite of DWD (cont'd)



GME, COSMO-EU, COSMO-DE
(colours show orography)



ICON

Treatment of SBL Turbulence in DWD Models

- **GME**: algebraic turbulence scheme, quasi-conservative moist variables, RH SGS cloud scheme, Ri-dependent stability functions

- **COSMO-EU/DE**: one-equation (TKE) turbulence scheme, statistical SGS cloud scheme (similar to Sommeria & Deardorff 1977), Mellor-Yamada-type stability functions, parameterization of horizontal heterogeneity at the sub-grid scales ("SGS circulation terms", Raschendorfer 1999, 2001)

- **ICON** (work in progress): TKE scheme (c/o Matthias Raschendorfer), EDMF scheme (c/o Martin Köhler), statistical SGS cloud scheme

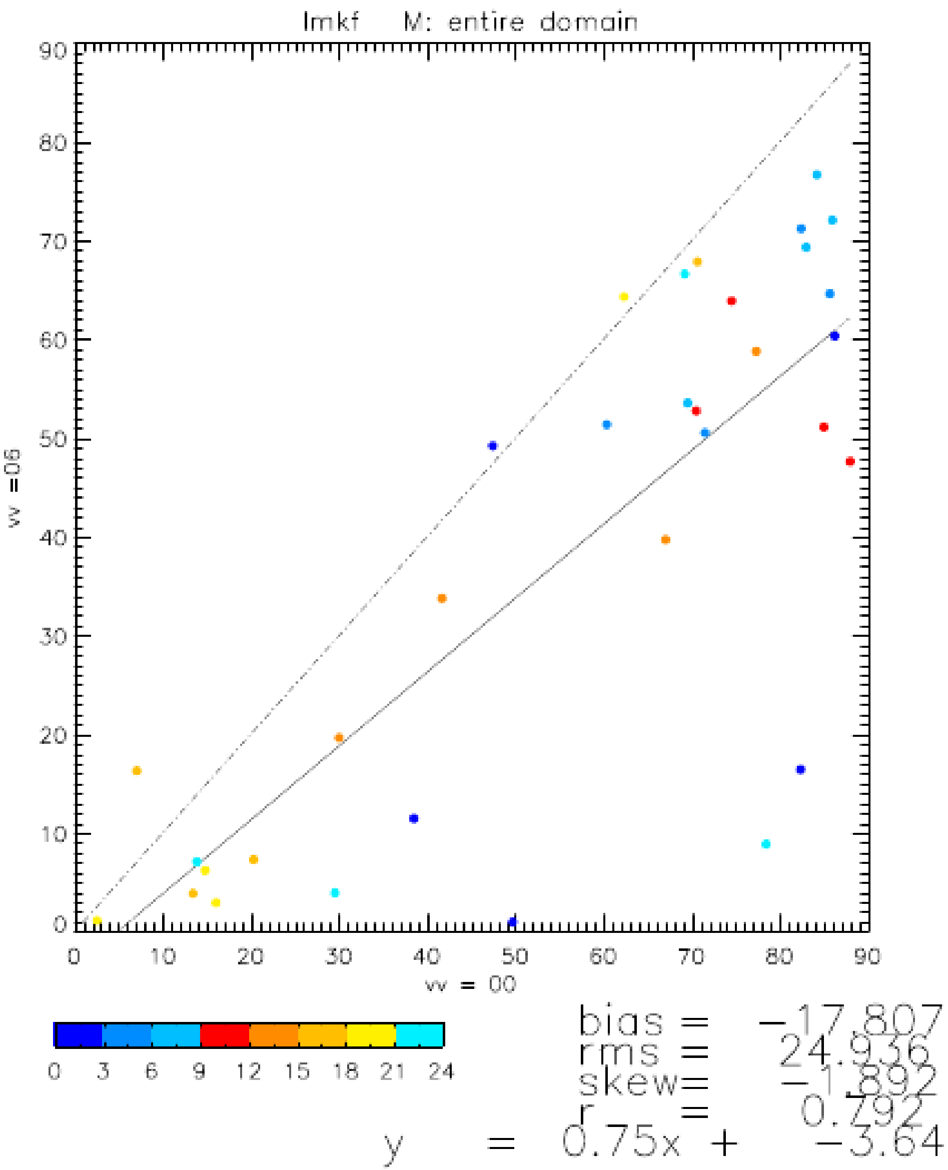
SBL in DWD Models (cont'd)

- Parameterization of SGS horizontal heterogeneity ("SGS circulation terms", see Raschendorfer 1999, 2001, for details): accounts for enhanced mixing in the SBL over horizontally-heterogeneous surfaces (cf. results of PS & DM)

- Minimum vertical diffusion coefficient K_{min}=1 m^2/s

- No tile approach, every grid box is characterised by a single soil type

SBL in DWD Models (cont'd)

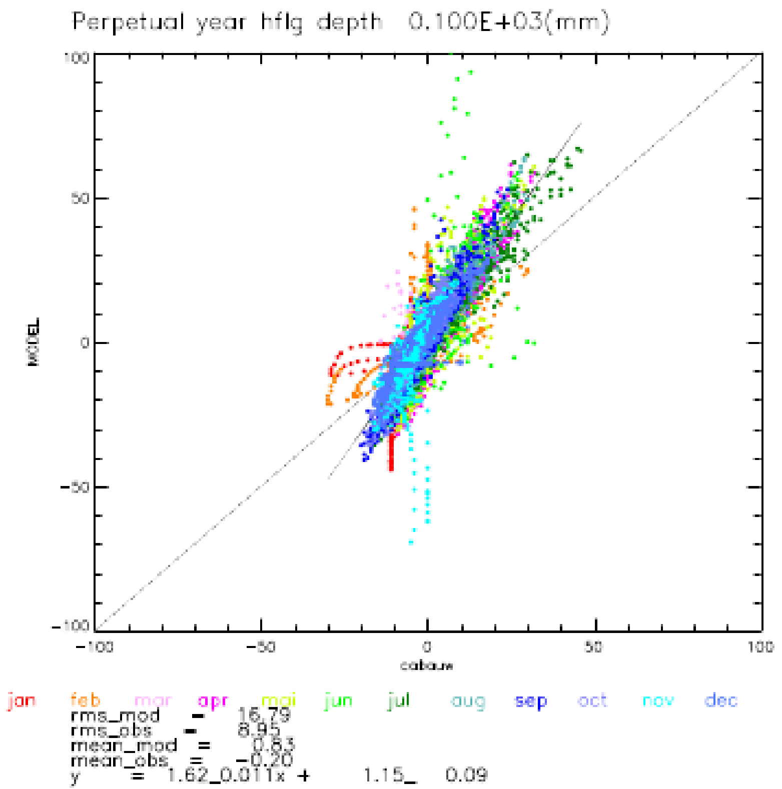


Low-level cloud cover in COSMO-DE, 9-13 October 2008, River Main area.
vv=00 – analysis,
vv=06 – 6h forecast for the same target time.

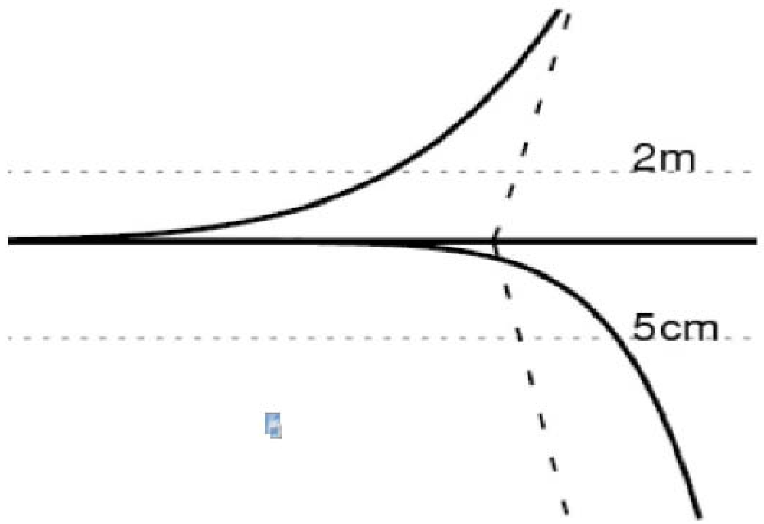
COSMO-DE tends to destroy clouds (i.a. K_{min} is too high).

Patrick Volker, DWD

SBL in DWD Models (cont'd)



Soil heat flux at 10 cm below the surface from the observations at Cabauw (abscissa) and from the stand-alone perpetual-year COSMO run (ordinate).



Schematic representation of the temperature profile in the near-surface layers of air and soil. Solid curve – observations, dashed line – model.

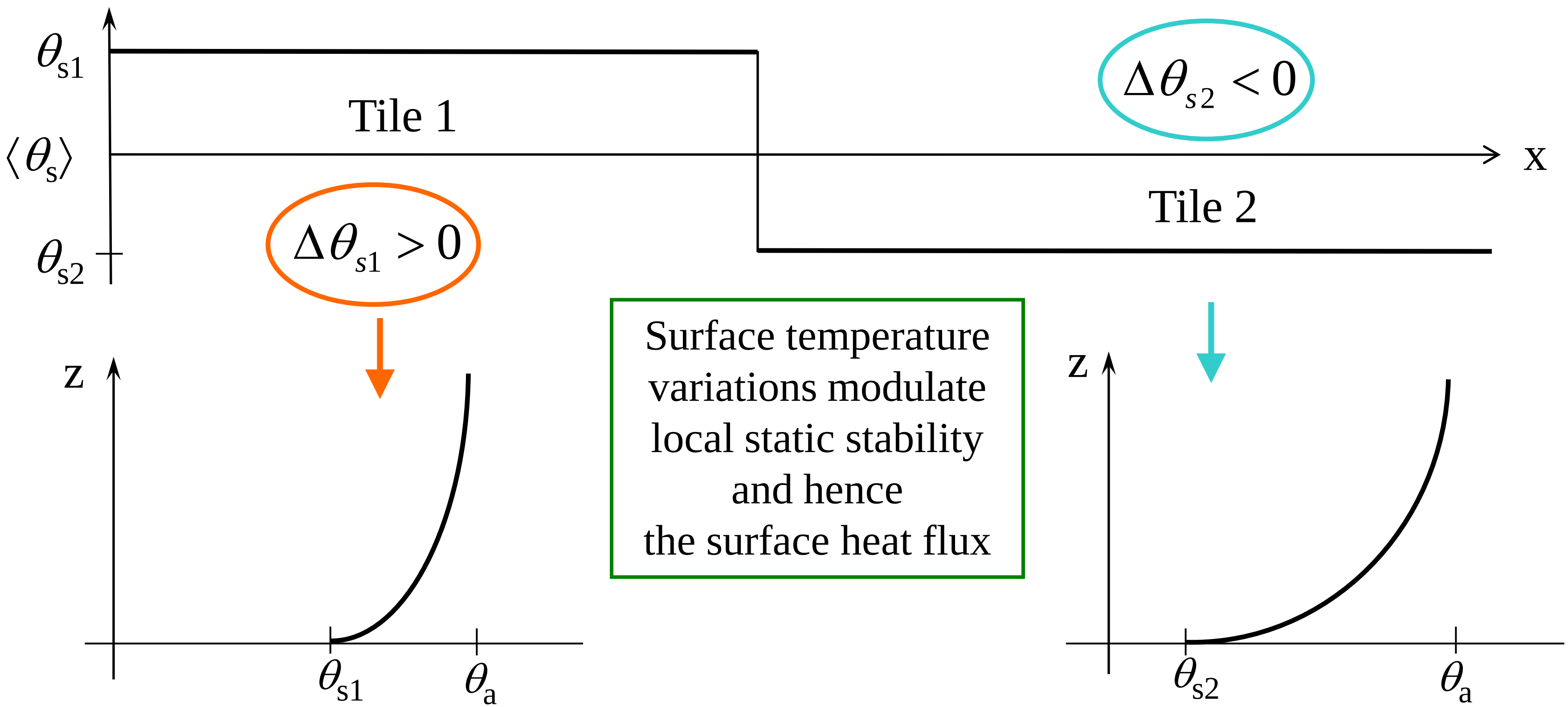
Soil is too diffusive, surface temperature is too high.

Can We Improve the Situation Using LES Results?

In order to describe enhanced mixing in heterogeneous SBL, an increased $<\theta'^2>$ at the surface should be accounted for.

- **Elegant way**: modify the surface-layer flux-profile relationships. Difficult – not for nothing are the Monin-Obukhov surface-layer similarity relations used for more than 50 years without any noticeable modification!
- **Less elegant way**: use a tile approach, where several parts with different surface temperatures are considered within an atmospheric model grid box.

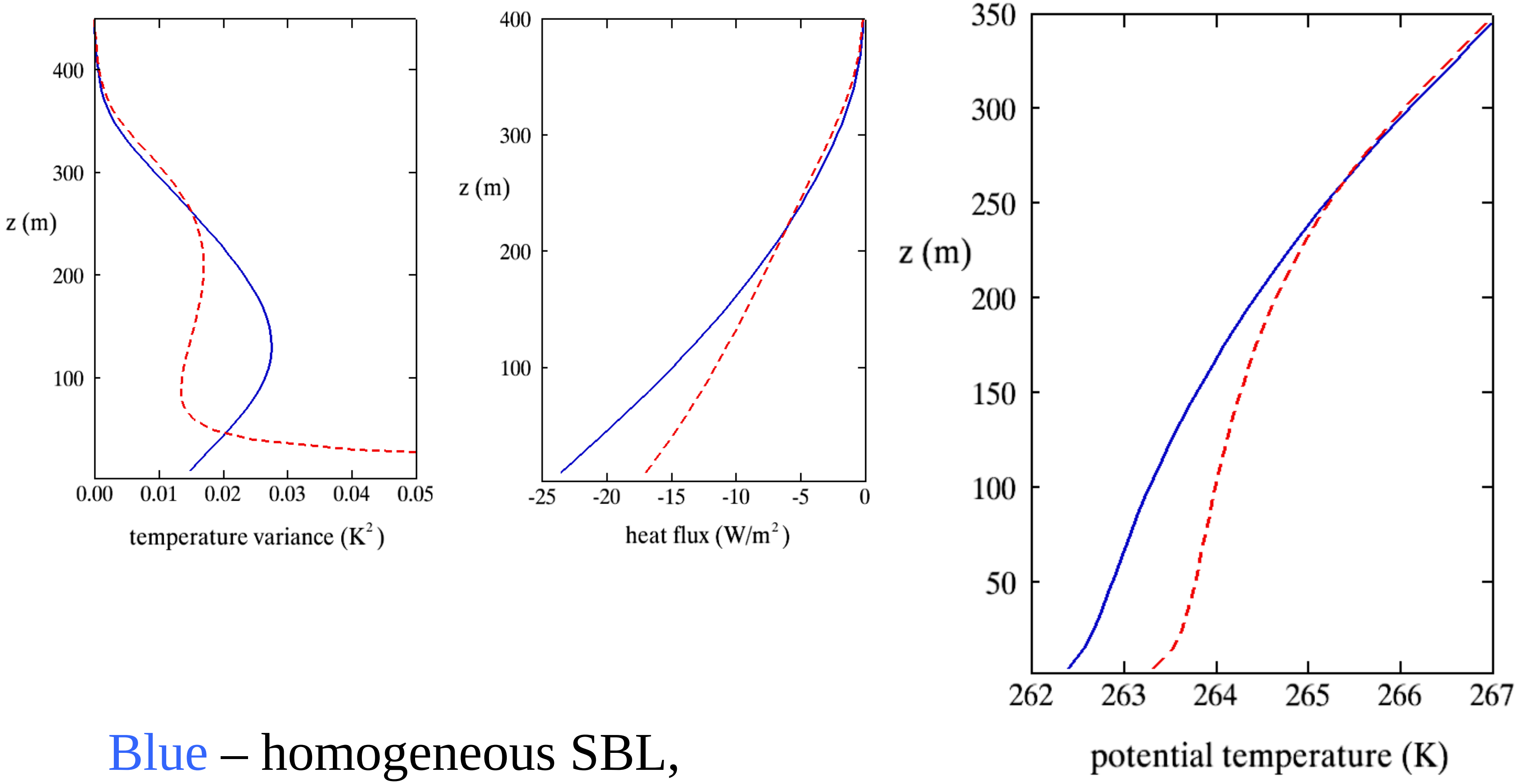
Coupling of the TKE-Scalar Variance Scheme with the Tiled Surface Scheme



$$\Delta\left(\overline{w'\theta'}\right)_1 > 0 \;\rightarrow\; \overline{w'\theta'^2} = \frac{1}{2}\left[\Delta\theta_{s1}\Delta\left(\overline{w'\theta'}\right)_1 + \Delta\theta_{s2}\Delta\left(\overline{w'\theta'}\right)_2\right] \neq 0 \;\leftarrow\; \Delta\left(\overline{w'\theta'}\right)_2 < 0$$

→ net gain/loss of $<\theta'^2>$ due to non-zero third-order transport term!

Tiled TKE-Temperature Variance Model: Results



Blue – homogeneous SBL,

red – heterogeneous SBL.

SBL in DWD Models: Work in Progress

- Implementation of tile approach (Ekaterina Machulskaya, Jürgen Helmert)

(i) only a few tiles are considered but the tiles with the maximum difference in terms of their thermal inertia must be included,

(ii) individual profiles of temperature (and water content) are considered for each tile,

(iii) SGS inland water is crucial (treated with FLake, http://lakemodel.net)

- Further development of parameterization of "circulation terms" (Matthias Raschendorfer)

- Development of the TKE-Scalar Variance scheme, incl. prognostic treatment and third-order transport of scalar variances, and coupling with tiled surface scheme (COSMO UTCS project)

- Development of an extended statistical SGS cloud scheme to account for the skewness of scalars (DWD, MPI-M, University of Hannover)

Conclusions and Outlook

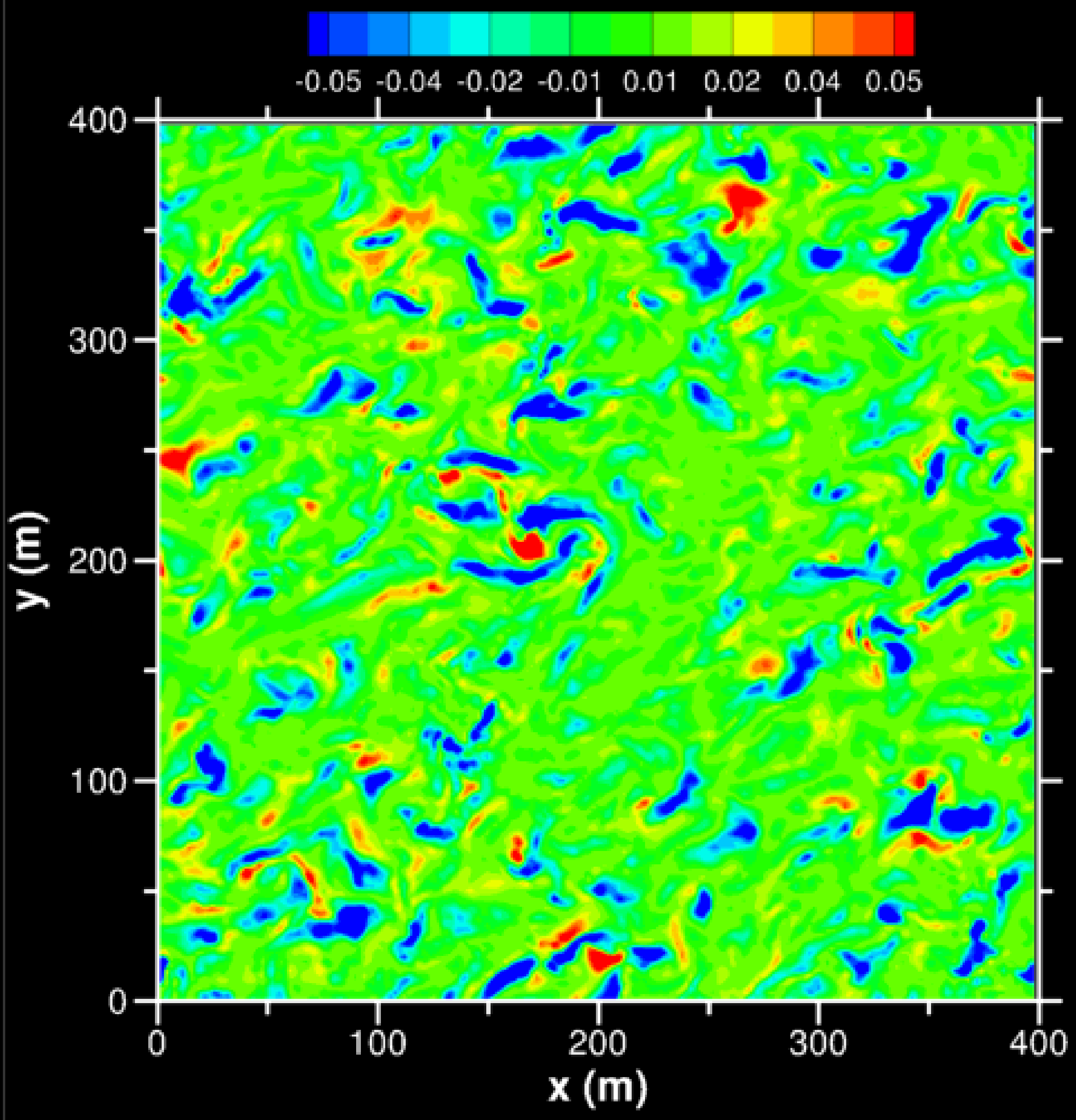
- LES results suggests plausible explanation of enhanced mixing in horizontally-heterogeneous SBL
- Turbulent transport of temperature variances (third-order term $<w'\theta'^2>$ in the $<\theta'^2>$ budget) is an important point
- Ways to improve SBL parameterisations are outlined

- Simulations of strongly stable PBL (PS & DM)
- Comprehensive analysis of pressure-scalar and pressure-velocity covariances in the second-moment budgets (DM & PS)
- Development of improved SBL parameterisations, e.g. tile approach, treatment of scalar variances (DWD, NCAR, etc.)

Thanks for your attention!

Acknowledgements: Vittorio Canuto, Sergey Danilov, Evgeni Fedorovich, Vladimir Gryanik, Donald Lenschow, Ekaterina Machulskaya, Chin-Hoh Moeng, Ned Patton, Matthias Raschendorfer, Bodo Ritter, Patrick Volker, Jeffrey Weil.
The work was partially supported by the NCAR Geophysical Turbulence Program and by the European Commission through the COST Action ES0905.

$w'\theta'$ at 12.5 m above the surface



Unused

Scalar Variance Budget Derived from LES Data

Resolved-scale scalar variance

$$\frac{1}{2}\frac{d\left\langle \bar{f}''^2\right\rangle}{dt} = -\left\langle \bar{u}_i''\bar{f}''\right\rangle \frac{\partial\left\langle \bar{f}\right\rangle}{\partial x_i} - \frac{1}{2}\frac{\partial}{\partial x_i}\left\langle \bar{u}_i''\bar{f}''^2\right\rangle - \left\langle \bar{f}''\frac{\partial \overline{u_i' f'}''}{\partial x_i}\right\rangle$$

Spectral transfer through the filter scale

SGS scalar variance

$$\frac{1}{2}\frac{d\left\langle \overline{f'^2}\right\rangle}{dt} = -\left\langle \overline{u_i' f'}\right\rangle \frac{\partial\left\langle \bar{f}\right\rangle}{\partial x_i} - \frac{1}{2}\frac{\partial}{\partial x_i}\left(2\left\langle \bar{f}''\overline{u_i' f'}''\right\rangle + \left\langle \bar{u}_i''\overline{f'^2}''\right\rangle + \left\langle \overline{u_i' f'^2}\right\rangle\right) - \varepsilon_{f^2} + \left\langle \bar{f}''\frac{\partial \overline{u_i' f'}''}{\partial x_i}\right\rangle,$$

Implications

Good News

- Analysis of LES results suggests plausible explanation of enhanced mixing in horizontally-heterogeneous SBL – <u>we understand more</u> (increased <θ'^2> near the surface, i.a. due to non-zero <$w'\theta'^2$>, is a key point)

Bad News

- Major increase of <θ'^2> in heterogeneous SBL occurs near the surface – <u>difficult to parameterise</u>

Tiled TKE-Temperature Variance Closure Model
(with Ekaterina Machulskaya, DWD)

- Prognostic equations for TKE and for the temperature variances including third-order transport
- Algebraic (diagnostic) formulations for temperature flux, for the Reynolds-stress components, and for turbulence length scale
- Tile approach where different tiles have different surface temperature
- Surface fluxes are computed as weighted means of fluxes over individual tiles
- $<w'\theta'^2>$ is non-zero at the surface in heterogeneous SBL
- Input parameters of numerical experiments are similar to LES (except for piece-wise vs. sinusoidally varying surface temperature)

■ ■ ■

Large-Eddy Simulations

Boundary-layer flows over temperature-homogeneous vs. temperature-heterogeneous surface

- LES code: Moeng (1984), Moeng and Wyngaard (1998), Sullivan et al. (1994, 1996), Sullivan and Patton (2008, 2011).
- Domain: 400×400×400 m, 200×200×192 mesh points, 2 m mesh size.
- Geostrophic wind: (8,0) $m \cdot s^{-1}$, Coriolis parameter: $1.39 \cdot 10^{-4}$ s^{-1}, temperature gradient above the PBL: 10^{-2} $K \cdot m^{-1}$.
- Boundary conditions: doubly periodic in *x* and *y* horizontal directions, the Monin-Obukhov surface-layer similarity relations are applied point-by-point.
- Initial temperature profile: mixed layer of depth 100 m and temperature 265 K, temperature increases linearly aloft at a rate 10^{-2} $K \cdot m^{-1}$.
- In homogeneous case, a constant surface cooling rate of -0.375 $K \cdot hr^{-1}$ over 8 hrs. In heterogeneous case, the surface cooling rate varies sinusoidally in the streamwise direction leading to a surface temperature difference of 6 K between the warm and the cold stripes (cf. Stoll and Porté-Agel 2009). Following this initial 8 hr period, a constant surface cooling rate of -0.375 $K \cdot hr^{-1}$ in both cases.

Analysis of LES Data

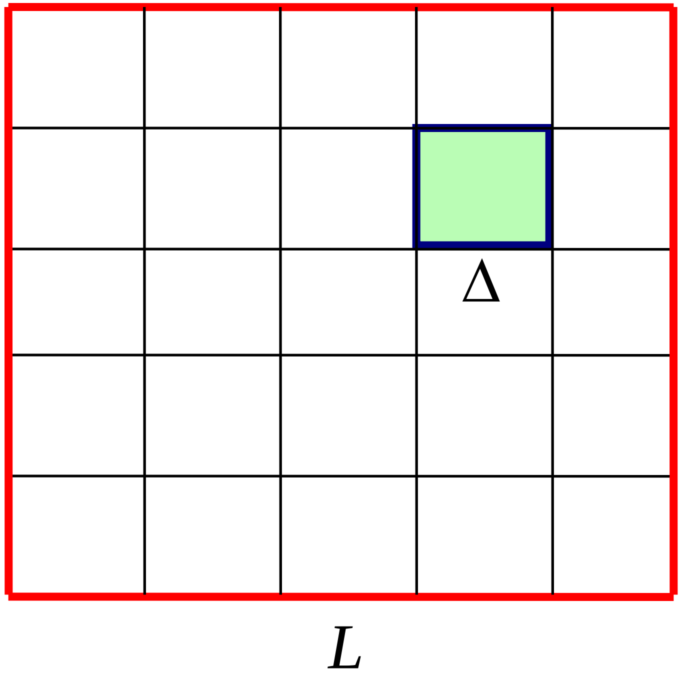
- In order to obtain approximations to ensemble-mean quantities, the LES data are averaged over horizontal planes and the resulting profiles are then averaged over more than 8000 time steps (the number of samples varies between cases). The sampling time covers the last 1.75 hours of simulations.

- Mean fields, second-order and third-order moments
- Budgets of TKE, of the temperature variance and of the temperature flux with due regard for SGS contributions (important in SBL even at high resolution)
- Implications for SBL turbulence parameterisations

Estimation of Total Variances

A triple decomposition is applied, using (i) a low-pass filter (overbar) whose characteristic horizontal scale Δ is much less than the domain size *L*, and (ii) a horizontal averaging operator over *L* (angle brackets).

A fluctuating quantity *f* may then be represented as a sum of the horizontal mean filtered part, a deviation of the filtered quantity from the horizontal mean (double prime), and a sub-filter fluctuation (prime):

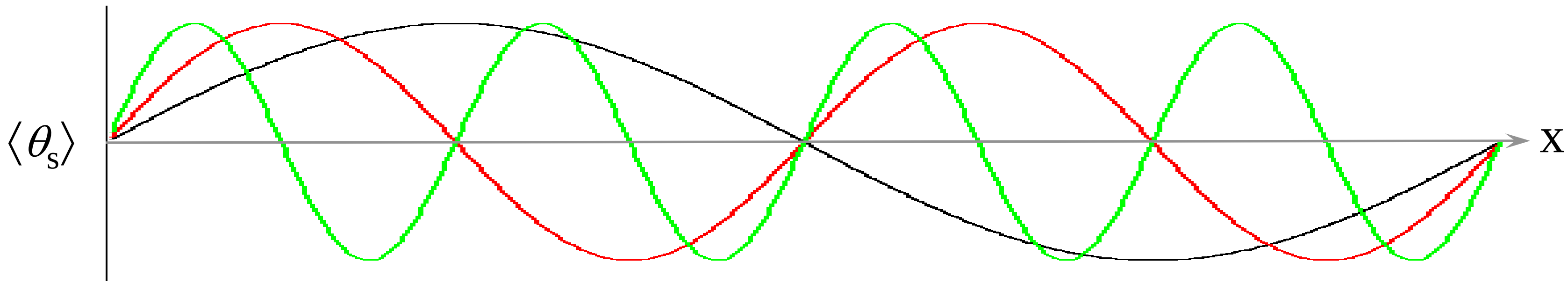


$$f = \langle \bar{f} \rangle + \bar{f}'' + f'$$

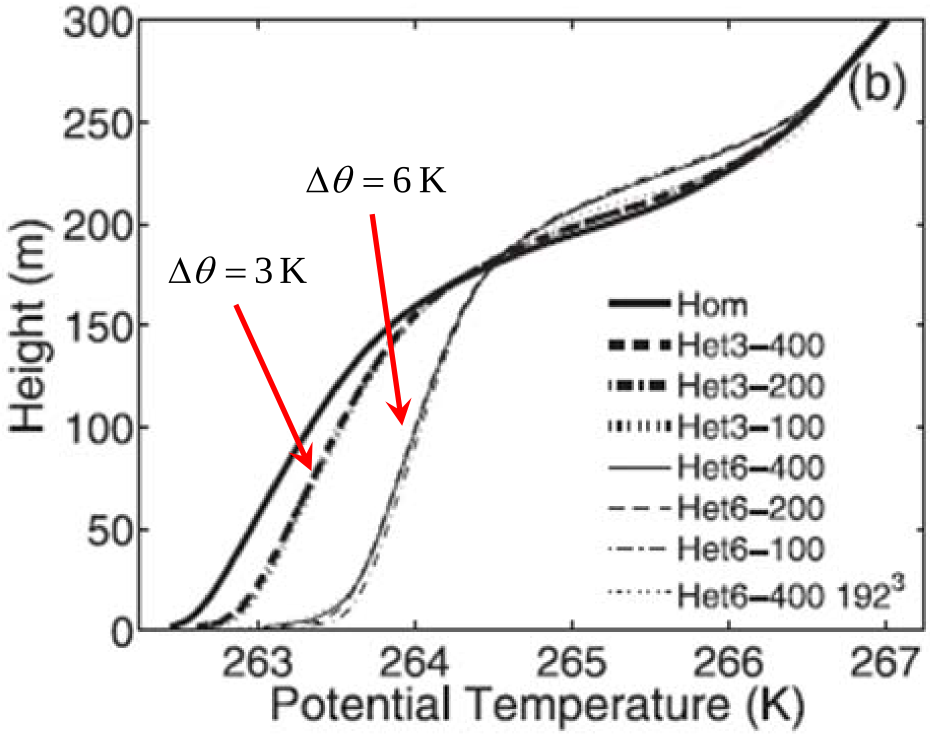
Then

$$total\ variance = \langle \bar{f}''^2 \rangle + \langle \overline{f'^2} \rangle$$

Surface Temperature (cont'd)



The number of stripes does not affect the results, what matters is the temperature difference between warm and cold stripes (Stoll and Porté-Agel 2009)



Estimation of Total Variances

We apply a triple decomposition, using (i) a low-pass filter whose characteristic horizontal scale, Δ, is much less than the domain size, *L*, and (ii) a horizontal averaging operator over *L*. A fluctuating quantity *f* may then be represented as a sum of the horizontal mean filtered part, a deviation of the filtered quantity from the horizontal mean, and a sub-filter fluctuation,

$$f = \left\langle \bar{f} \right\rangle + \bar{f}'' + f',$$

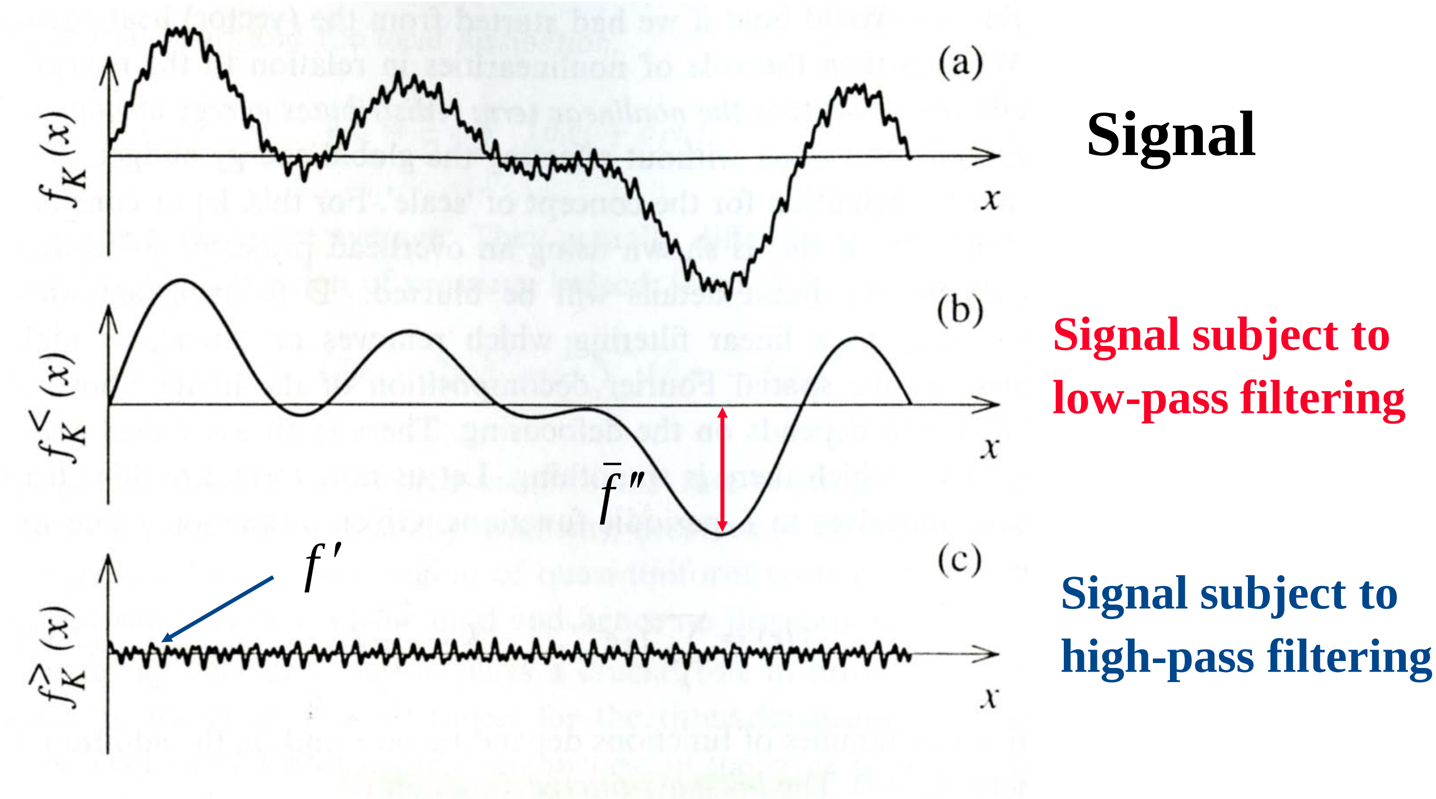
where an overbar denotes a low-pass filtered quantity, and a prime denotes a deviation therefrom. Angle brackets denote averaging over the horizontal, and a double prime denotes a fluctuation about a horizontal mean.

There is nothing really new in it, cf.

- Mean flow-wave-turbulence decomposition (Hussein and Reynolds 1970, 1972, Reynolds and Hussein 1972)
- A procedure used in LES studies to compute (approximations to) ensemble-mean statistical moments as a sum of resolved scale and sub-grid scale contributions (e.g. Brown 1995, Mironov et al. 2000, Mironov 2001)
- Energy budget scale-by-scale (Frisch 1995, section 2.4)

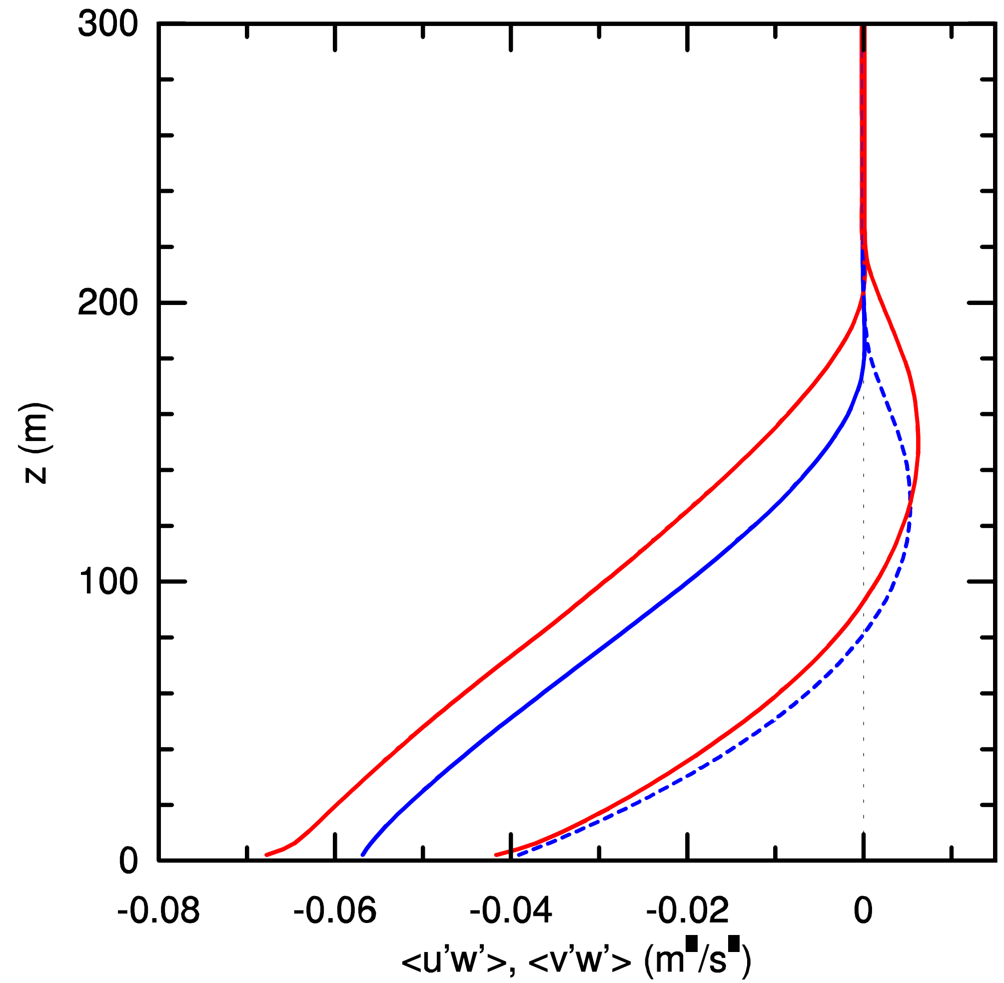
Estimation of Total Variances (cont'd)

Low-pass filtered and high-pass filtered quantity



$$total\ variance = \left\langle \bar{f}''^2 \right\rangle + \left\langle \overline{f'^2} \right\rangle$$

Components of Momentum Flux



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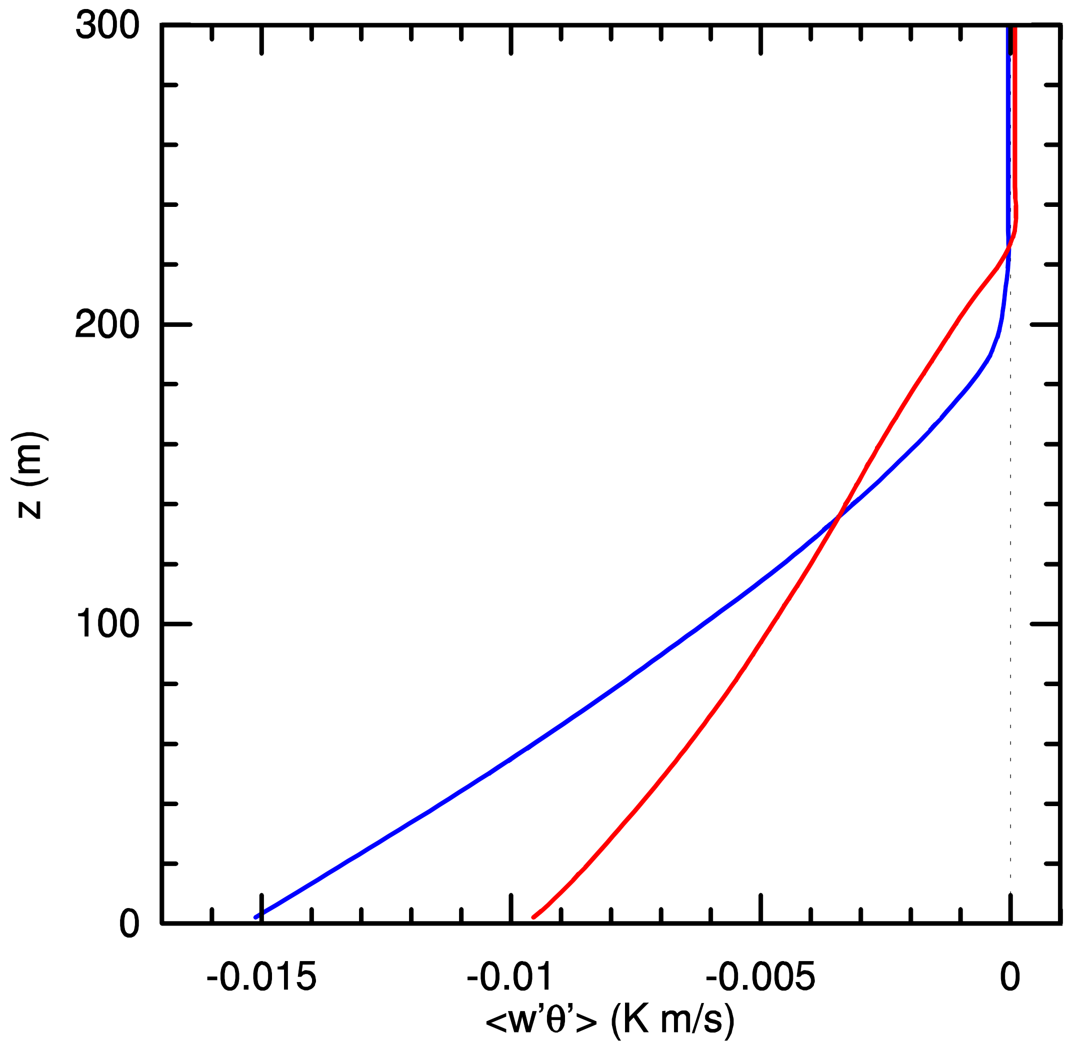
Vertical Temperature Flux



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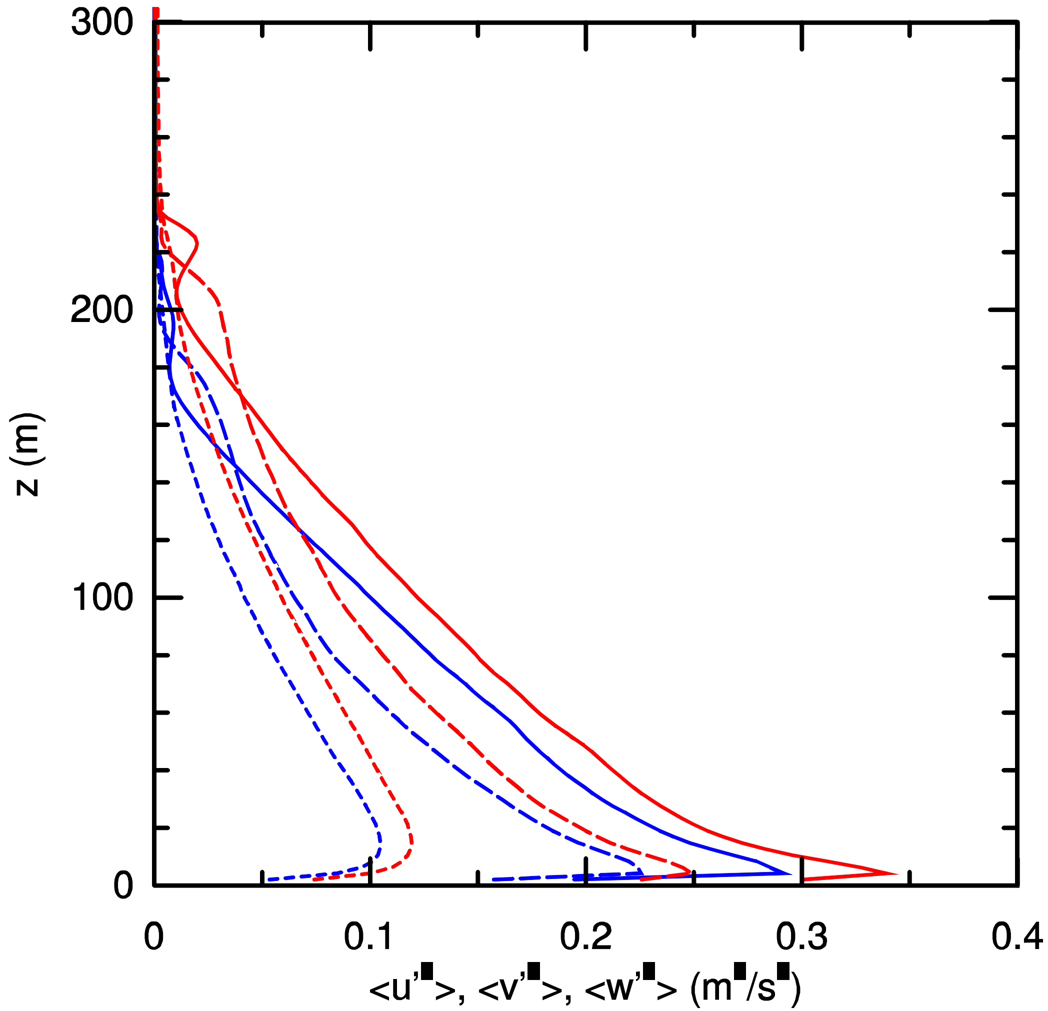
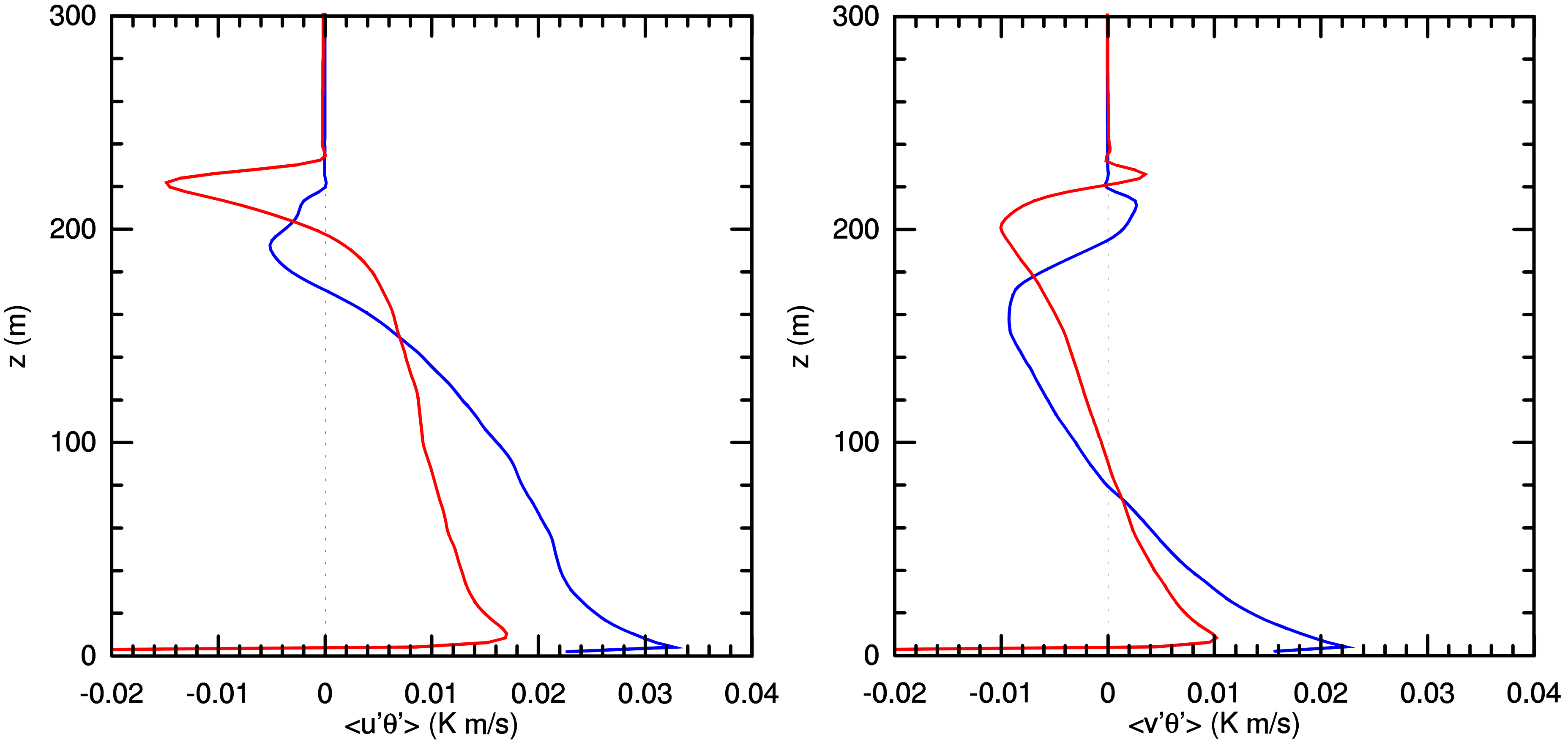
Velocity Variances



Blue – homogeneous SBL, red – heterogeneous SBL.

Short-dashed – $<w'^2>$, long-dashed – $<v'^2>$, solid – $<u'^2>$.

Streamwise and Spanwise Temperature Flux



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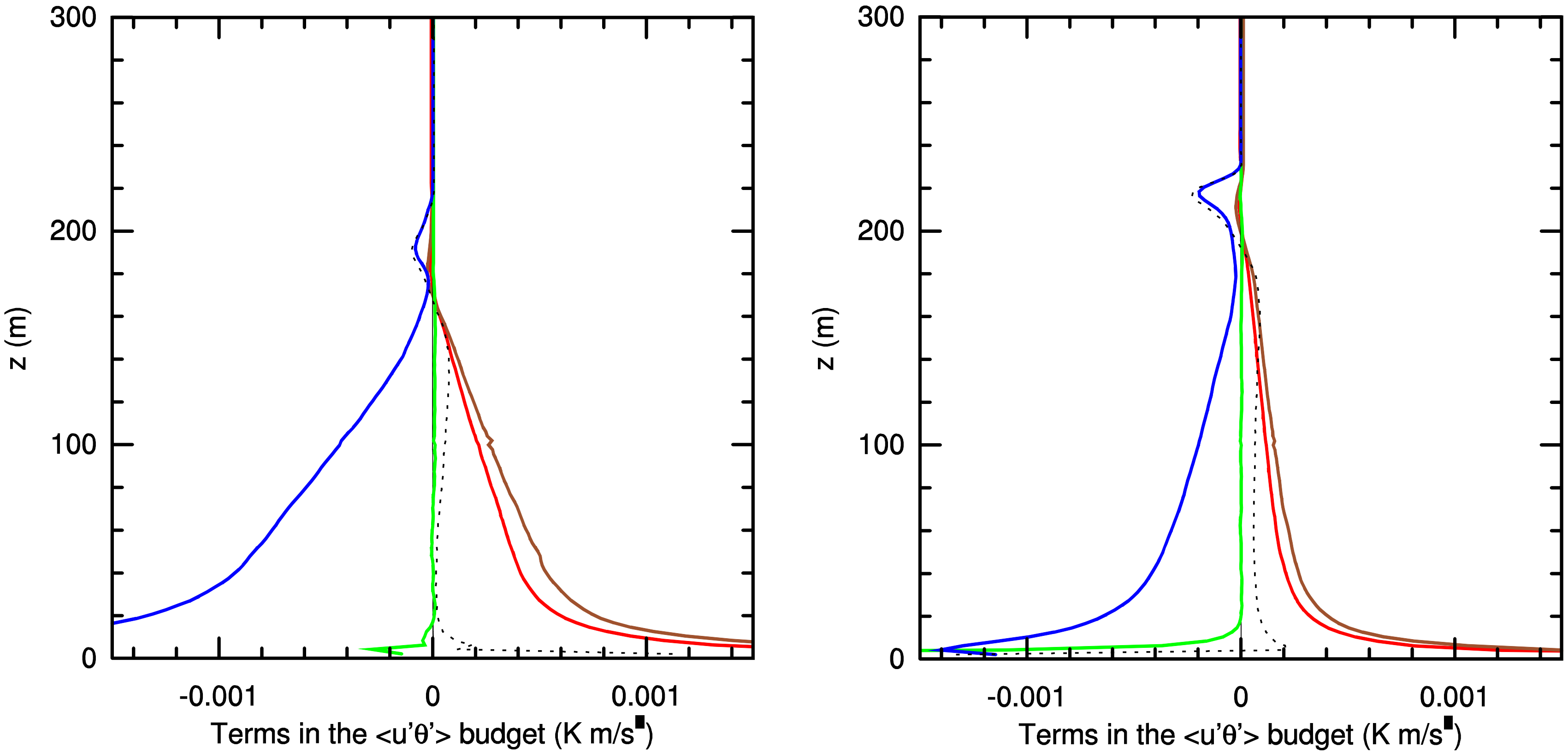
Streamwise Temperature Flux Budget

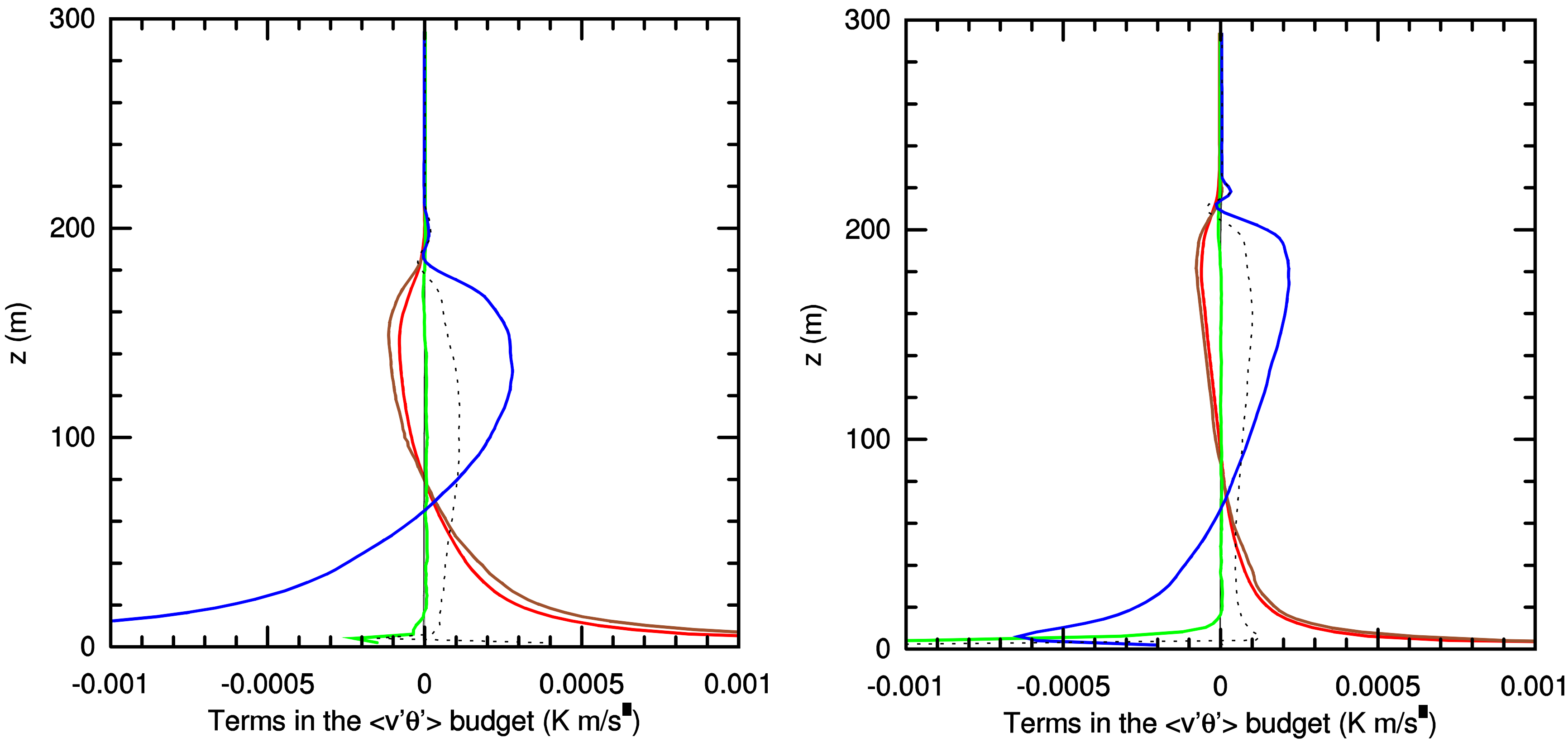


Left panel – homogeneous SBL, right panel – heterogeneous SBL.

Red – mean temperature gradient, **brown** – mean velocity shear, **blue** – pressure gradient-temperature covariance, **green** – third-order transport, thin dotted black – tendency.

$$\frac{\partial \overline{u'\theta'}}{\partial t} = -\overline{u'w'}\frac{\partial \overline{\theta}}{\partial z} - \overline{w'\theta'}\frac{\partial \overline{u}}{\partial z} - \frac{\partial}{\partial z}\overline{w'u'\theta'} - \overline{\theta'\frac{\partial p'}{\partial x}}$$

Spanwise Temperature Flux Budget



Left panel – homogeneous SBL, right panel – heterogeneous SBL.

Red – mean temperature gradient, **brown** – mean velocity shear, **blue** – pressure gradient-temperature covariance, **green** – third-order transport, thin dotted black – tendency.

$$\frac{\partial \overline{v'\theta'}}{\partial t} = -\overline{v'w'}\frac{\partial \overline{\theta}}{\partial z} - \overline{w'\theta'}\frac{\partial \overline{v}}{\partial z} - \frac{\partial}{\partial z}\overline{w'v'\theta'} - \overline{\theta'\frac{\partial p'}{\partial y}}$$

Vertical Temperature Flux Budget

$$-\overline{w'^2}\frac{\partial\overline{\theta}}{\partial z} + g\alpha\overline{\theta'^2} - \frac{\partial}{\partial z}\overline{w'^2\theta'} - \overline{\theta'\frac{\partial p'}{\partial z}} \approx 0$$

Mean-gradient production of downward (negative) temperature flux

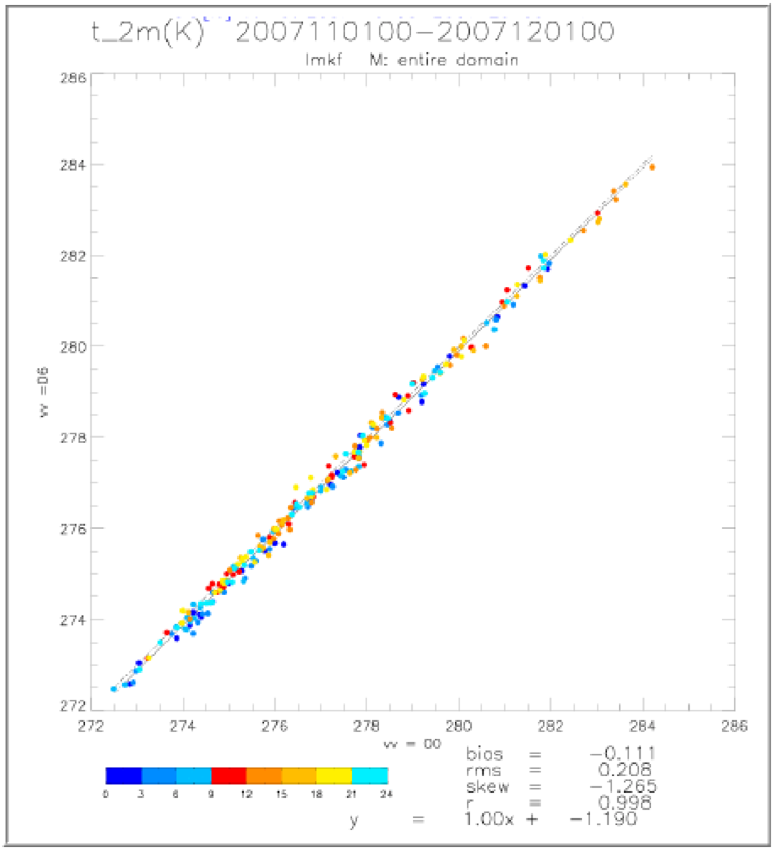
Buoyancy production of upward (positive) temperature flux

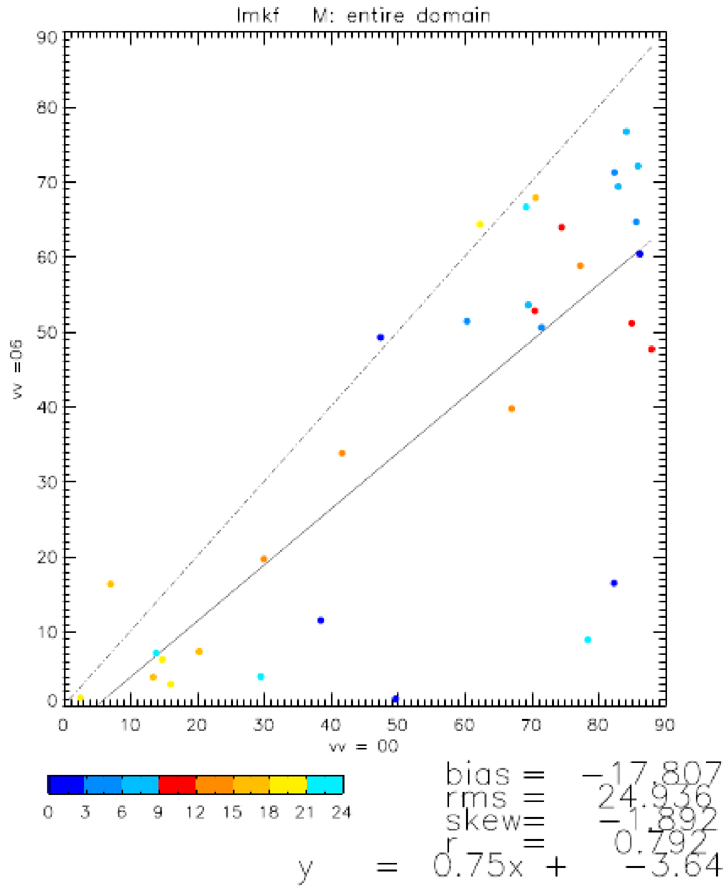
Transport (small)

Flux destruction (return-to-isotropy)

Increased $\langle\theta'^2\rangle$ near the surface in horizontally-heterogeneous SBL $\Rightarrow$ reduced magnitude of downward temperature flux

SBL in DWD Models (cont’d)





Left – T2m, November 2007, entire COSMO-DE domain;
right – low-level cloud cover 9-13 October 2008, River Main area.
vv=00 – analysis, vv=06 – 6h forecast for the same target time.
COSMO-DE tends to destroy clouds (i.a. K_{min}=is too high).

Tiled TKE-Temperature Variance Closure Model (cont'd)

TKE

$$\frac{\partial e}{\partial t} = -\left(\overline{w'u'} \frac{\partial \overline{u}}{\partial z} + \overline{w'v'} \frac{\partial \overline{v}}{\partial z} \right) + g\alpha \overline{w'\theta'} - \frac{\partial}{\partial z} \left(\frac{1}{2} \overline{w'u_i'^2} + \overline{w'p'} \right) - \varepsilon$$

Temperature variance

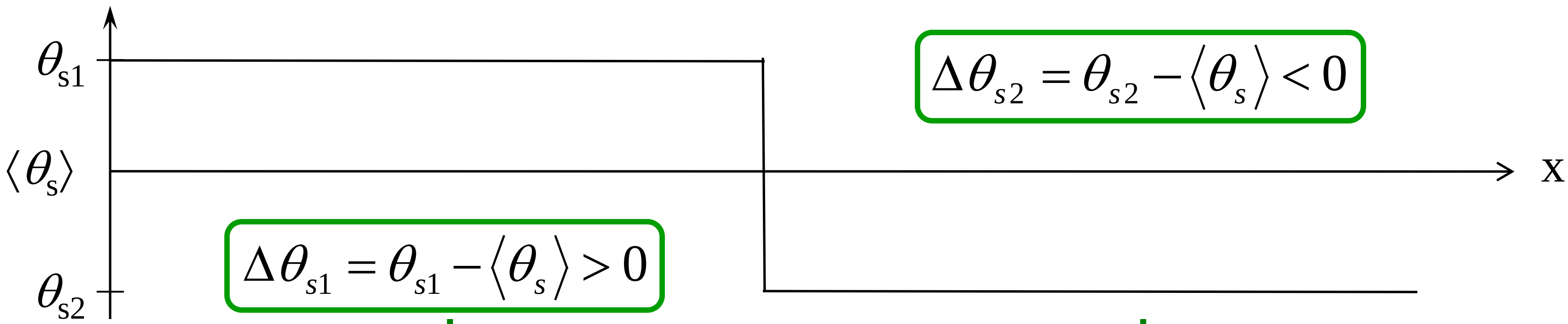
$$\frac{1}{2} \frac{\partial \overline{\theta'^2}}{\partial t} = -\overline{w'\theta'} \frac{\partial \overline{\theta}}{\partial z} - \frac{1}{2} \frac{\partial}{\partial z} \overline{w'\theta'^2} - \varepsilon_\theta$$

Temperature (heat) flux, momentum flux, turbulence time (length) scale

$$\overline{w'\theta'} = -C_h \tau e \frac{\partial \overline{\theta}}{\partial z} + (1 - C_b) \tau g \alpha \overline{\theta'^2}, \quad \overline{u'w'} = -C_m \tau e \frac{\partial \overline{u}}{\partial z}, \quad \overline{v'w'} = -C_m \tau e \frac{\partial \overline{v}}{\partial z},$$

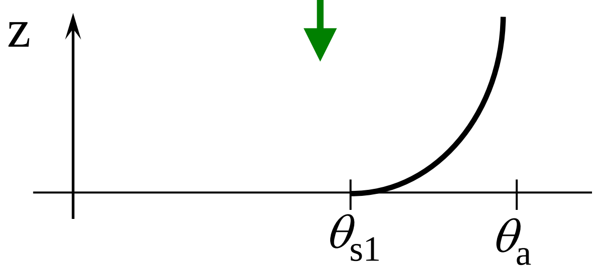
$$\tau = l e^{-1/2}, \quad l = l(z, h, e, N, \ldots)$$

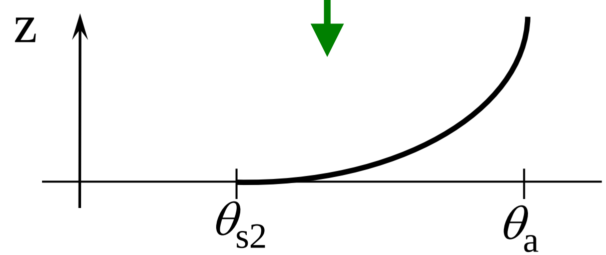
Tiled TKE-Temperature Variance Closure Model (cont'd)



$$\Delta\theta_{s2} = \theta_{s2} - \langle\theta_s\rangle < 0$$

$$\Delta\theta_{s1} = \theta_{s1} - \langle\theta_s\rangle > 0$$





$$\Delta\left(\overline{w'\theta'}\right)_1 = \left(\overline{w'\theta'}\right)_1 - \left\langle\overline{w'\theta'}\right\rangle > 0$$

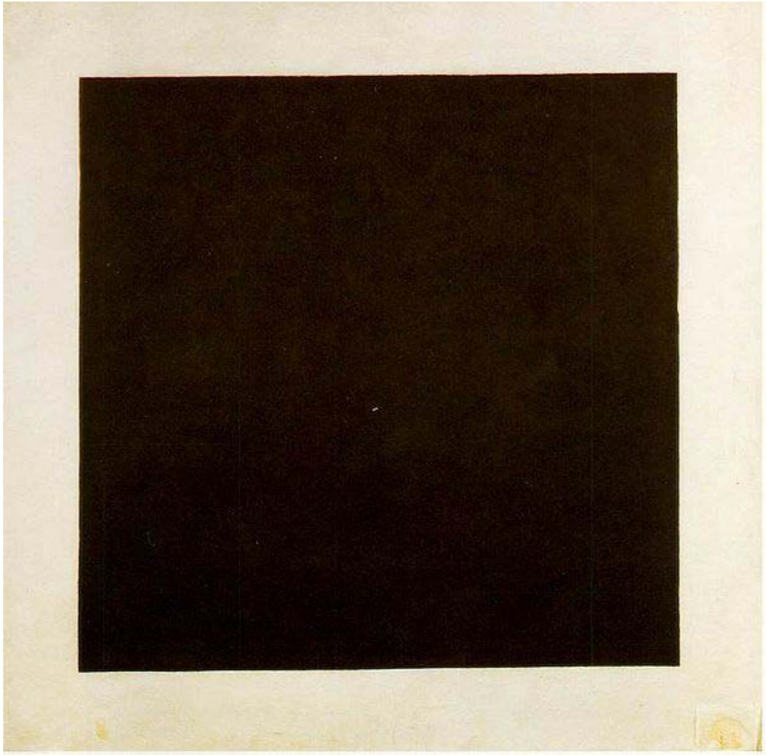
$$\Delta\left(\overline{w'\theta'}\right)_2 = \left(\overline{w'\theta'}\right)_2 - \left\langle\overline{w'\theta'}\right\rangle < 0$$

$$\overline{\overline{w'\theta'}}^{2} = \left\langle \bar{\theta}''\overline{w'\theta'}'' \right\rangle = \frac{1}{2}\left[\Delta\theta_{s1}\Delta\left(\overline{w'\theta'}\right)_1 + \Delta\theta_{s2}\Delta\left(\overline{w'\theta'}\right)_2\right] \neq 0$$

Low boundary condition for $<\theta'^2>$

■ ■ ■

$w'\theta'$ at 12.5 m above the surface



Kasimir Malevich, Black Square, 1915

(as a conceptual model of nocturnal SBL)

- Ad hoc tuning devices like "minimum diffusion coefficients" do not help much (they are often detrimental for the NWP/climate model performance)

Ensemble-Mean Second-Moment Budget Equations

TKE

$$\frac{\partial e}{\partial t} = -\left(\overline{w'u'}\frac{\partial \overline{u}}{\partial z} + \overline{w'v'}\frac{\partial \overline{v}}{\partial z}\right) + g\alpha\overline{w'\theta'} - \frac{\partial}{\partial z}\left(\frac{1}{2}\overline{w'u_i'^2} + \overline{w'p'}\right) - \varepsilon$$

Temperature variance

$$\frac{1}{2}\frac{\partial \overline{\theta'^2}}{\partial t} = -\overline{w'\theta'}\frac{\partial \overline{\theta}}{\partial z} - \frac{1}{2}\frac{\partial}{\partial z}\overline{w'\theta'^2} - \varepsilon_\theta$$

Temperature (heat) flux

$$\frac{\partial \overline{w'\theta'}}{\partial t} = -\overline{w'^2}\frac{\partial \overline{\theta}}{\partial z} + g\alpha\overline{\theta'^2} - \frac{\partial}{\partial z}\overline{w'^2\theta'} - \overline{\theta'\frac{\partial p'}{\partial z}}$$

Ensemble-Mean Second-Moment Budget Equations

TKE

$$\frac{1}{2}\left(\frac{\partial}{\partial t}+\bar{u}_k\frac{\partial}{\partial x_k}\right)\overline{u_i'^2}=-\overline{u_i'u_k'}\frac{\partial\overline{u_i}}{\partial x_k}+g_i\alpha\overline{u_i'\theta'}-\frac{\partial}{\partial x_k}\left(\frac{1}{2}\overline{u_k'u_i'^2}+\overline{u_k'p'}\right)-\varepsilon$$

Temperature variance

$$\frac{1}{2}\left(\frac{\partial}{\partial t}+\bar{u}_i\frac{\partial}{\partial x_i}\right)\overline{\theta'^2}=-\overline{u_i'\theta'}\frac{\partial\bar{\theta}}{\partial x_i}-\frac{1}{2}\frac{\partial}{\partial x_i}\overline{u_i'\theta'^2}-\varepsilon_\theta$$

Temperature (heat) flux

$$\left(\frac{\partial}{\partial t}+\bar{u}_k\frac{\partial}{\partial x_k}\right)\overline{u_i'\theta'}=-\overline{u_k'\theta'}\frac{\partial\bar{u}_i}{\partial x_k}-\overline{u_i'u_k'}\frac{\partial\bar{\theta}}{\partial x_k}+g_i\alpha\overline{\theta'^2}-2\varepsilon_{ijk}\Omega_j\overline{u_k'\theta'}-\frac{\partial}{\partial x_k}\overline{u_k'u_i'\theta'}-\overline{\theta'\frac{\partial p'}{\partial x_i}}$$

Ensemble-Mean Second-Moment Budget Equations

Reynolds stress

$$\left(\frac{\partial}{\partial t}+\bar{u}_k\frac{\partial}{\partial x_k}\right)\overline{u_i'u_j'}=-\left(\overline{u_i'u_k'}\frac{\partial \bar{u}_j}{\partial x_k}+\overline{u_i'u_k'}\frac{\partial \bar{u}_j}{\partial x_k}\right)-\left(g_i\alpha\overline{u_j'\theta'}+g_j\alpha\overline{u_i'\theta'}\right)-2\left(\varepsilon_{ilk}\Omega_l\overline{u_k'u_j'}+\varepsilon_{ilk}\Omega_l\overline{u_k'u_j'}\right)$$

$$-\left(\overline{u_i'\frac{\partial p'}{\partial x_j}}+\overline{u_j'\frac{\partial p'}{\partial x_i}}-\frac{2}{3}\delta_{ij}\frac{\partial}{\partial x_k}\overline{u_k'p'}\right)-\frac{\partial}{\partial x_k}\left(\overline{u_k'u_i'u_j'}+\frac{2}{3}\delta_{ij}\overline{u_k'p'}\right)-\varepsilon_{ij}$$

Temperature variance

$$\frac{1}{2}\left(\frac{\partial}{\partial t}+\bar{u}_i\frac{\partial}{\partial x_i}\right)\overline{\theta'^2}=-\overline{u_i'\theta'}\frac{\partial\bar{\theta}}{\partial x_i}-\frac{1}{2}\frac{\partial}{\partial x_i}\overline{u_i'\theta'^2}-\varepsilon_\theta$$

Temperature (heat) flux

$$\left(\frac{\partial}{\partial t}+\bar{u}_k\frac{\partial}{\partial x_k}\right)\overline{u_i'\theta'}=-\overline{u_k'\theta'}\frac{\partial\bar{u}_i}{\partial x_k}-\overline{u_i'u_k'}\frac{\partial\bar{\theta}}{\partial x_k}+g_i\alpha\overline{\theta'^2}-2\varepsilon_{ijk}\Omega_j\overline{u_k'\theta'}-\frac{\partial}{\partial x_k}\overline{u_k'u_i'\theta'}-\overline{\theta'\frac{\partial p'}{\partial x_i}}$$